Magnetic confinement of charged particles

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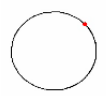

Figure 1: The motion of a charged particle in a constant uniform magnetic field is a simple circle in the plane perpendicular to $\vec{B}$.

Overview

This lecture covers the following topics:

- Motion in a straight uniform B-field (basic review)
- The guiding center approximation
- Motion including a static electric field, or gravity
- Motion in inhomogeneous and bent B-fields and the first adiabatic invariant
- Time varying fields
- The longitudinal invariant

The lecture begins with a basic review of the well-known case of a charged particle moving in a straight, uniform, static magnetic field. These equations form the basis from which the further derivations start. The basic spatial and temporal scales are derived for this simple case, and then more interesting cases are studied - first an added electric field, then a general force, and then the important case of a spatially inhomogeneous magnetic field is analyzed. The use of adiabatic invariants of the particle motion is introduced, and one sees how the identification and use of these invariants enables a relatively simple analysis of important phenomena. Time varying fields are also heuristically discussed towards the end of the lecture.

Charged particle motion in a straight magnetic field

We begin by reviewing the well-known text-book situation: A charged particle is confined by a straight uniform magnetic field and it feels no other forces. This results in a screw-like motion, which can be broken into a parallel free-streaming along the magnetic field, and a circular motion in the plane perpendicular to the magnetic field, shown in Figure 1. Although this is truly trivial, it is worth it here, since we will reuse the equations with minor (but important) modifications in the derivations that follow.
We start with Newton's 2^{nd} law and the Lorenz force:

$$\vec{F} = m\vec{a} = q\vec{v} \times \vec{B}$$

We choose our coordinate system such that the $\hat{z}$ direction lies along $\vec{B}$.

$$\vec{B} = B_0 \widehat{z}$$

We write Newton's 2^{nd} law coordinate by coordinate:

$$\vec{F} = m\vec{a} = q\vec{v} \times \vec{B} \Rightarrow$$

$$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y$$

$$\frac{dv_y}{dt} = -\frac{qB_0}{m} v_x$$

$$\frac{dv_z}{dt} = 0$$

It is clear from the equations that qB_0/m is an inverse time scale, so we give it its own symbol, ω_c, allowing us to write the equations:

$$\frac{dv_x}{dt} = \omega_c v_y$$

$$\frac{dv_y}{dt} = -\omega_c v_x$$

$$\frac{dv_z}{dt} = 0$$

$$\omega_c = \frac{qB_0}{m} - \text{sometimes } \omega_c = \frac{|q|B_0}{m}$$

If we can decouple the v_x and v_y equations, we can solve simple ordinary differential equations (ODEs). This is done by taking the time derivative in one of the equations and substituting the result into the other, as follows:

$$\frac{d}{dt}\left[\frac{dv_x}{dt} = \omega_c v_y\right]$$

$$\frac{dv_y}{dt} = -\omega_c = \omega_c$$

$$\omega_c = \frac{qB_0}{m} - \text{sometimes } \omega_c = \frac{|q|B_0}{m}$$

Now we eliminate v_y from the v_x equation by differentiation and substitution. The v_z equation is trivial to solve.

$$\left.\begin{aligned} \frac{d^2v_x}{dt^2} &= \omega_c \frac{dv_y}{dt} \\ \frac{dv_y}{dt} &= \omega_c v_x \end{aligned}\right\} \Rightarrow \frac{d^2v_x}{dt^2} = -\omega_c^2 v_x$$

$$\frac{dv_z}{dt} = 0 \Rightarrow v_z = constant = v_{\parallel}$$

$$\omega_c^2 = \left(\frac{qB_0}{m}\right)^2$$

At this point we eliminate v_y from the v_x equation by differentiation and substitution:

$$\left.\begin{aligned} \frac{dv_x}{dt} &= \omega_c v_y \\ \frac{dv_y}{dt} &= -\omega_c v_x \end{aligned}\right\} \Rightarrow \frac{d^2 v_x}{dt^2} = -\omega_c^2 v_x$$

We recognize the simple harmonic oscillator for v_x, and know the solution immediately. We then find v_y by differentiation.

$$v_x = v_\perp cos(\omega_c t + \delta)$$

$$v_y = -v_\perp sin(\omega_c t + \delta)$$

$$v_z = v_\parallel$$

We now integrate in time to come from velocity to position:

$$v_x = v_\perp cos(\omega_c t + \delta) \Rightarrow x = x_{gc} + \frac{v_\perp}{\omega_c} sin(\omega_c t + \partial)$$

$$v_y = -v_\perp sin(\omega_c t + \delta) \Rightarrow y = y_{gc} + \frac{v_\perp}{\omega_c} cos(\omega_c t + \partial)$$

$$v_z = v_\parallel \Rightarrow z = z_0 + v_\parallel t$$

We have introduced here the guiding center (x_{gc}, y_{gc}) around which the particle gyrates, and the Larmor radius :

$$r_L = \left|\frac{v_\perp}{\omega_c}\right| = \left|\frac{mv_\perp}{qB_0}\right|$$

which is the radius in the circular motion.

The guiding center approximation:

If the Larmor radius is on the order of the size of the confining field, then only collisionless orbits are confined, since, after a collision, the particle will have a new velocity vector and therefore a new guiding center. This will lie about one Larmor radius away from the original guiding center. If this new Larmor orbit intersects material walls or extends to regions of much lower B-field, the particle is not confined. Therefore, In order to confine charged particles magnetically for many collision times, the Larmor radius must be very small compared to the confining region.
When the Larmor radius is small compared to other relevant scale lengths, and the cyclotron frequency is large compared to other relevant frequencies,

one can derive analytic formulas for the time evolution of the guiding center (x_{gc}, y_{gc}, z), averaging over the gyration. Mathematically, what we require is:

$$r_L \ll B/\nabla B \text{ and } \omega_c \gg \frac{\partial B}{\partial t}\frac{1}{B}(^*)$$

The approximation of averaging over the gyration and deriving approximate equations for the the motion of the particle's guiding center is called the **guiding center approximation.**
The guiding center approximation is not just convenient, it is often necessary. We illustrate this with a simple example: For a 100 eV electron in a $B = 1$ T-field, the cyclotron frequency is large and the Larmor radius small:

$$\omega_c \approx 1.8 \times 10^{11} \text{ s}^{-1}$$
$$r_L \approx 3 \times 10^{-5} \text{ m}$$

This could for example be an electron in the edge region of a fusion experiment. In such experiments, the confinement time for the plasma on the closed flux surfaces is in the range 10^{-2} sec to several seconds, depending on various parameters. However, just to follow the electron for one microsecond, 10^{-6} sec, requires $> 10^6$ time steps if a simple numerical scheme is used. The computational effort is spent almost entirely on calculating the circular motion which was just derived analytically. Averaging over the gyromotion allows fast and accurate calculations of the motion of charged particles in a magnetic field. This is true for both analytic and numerical calculations. One may be surprised that it is more accurate numerically to make an approximation rather than solve the exact equations, but the numbers for our example show us why this could well be the case. Unavoidable numerical errors on the order of the machine accuracy may accumulate and lead to errors on the order of the results themselves, since, for a direct simulation for just a single second of simulation, 10^{12} computational steps are need. Even an algorithm with each time step done near machine accuracy of 10^{-12} may not be able to correctly predict the particle orbits for one second.

In the guiding center approximation that we will introduce step by step in the following, the gyrating particle is effectively replaced by a charged (q), massive (m) ring of current ($I = e\omega_c/2\pi$), with the ring center at the actual particle's gyrocenter. Of course, the cases of interest are now ones where one perturbs this idealized situation in order to bring in relevant deviations from the highly idealized situation of a straight uniform magnetic field with no other forces acting. effects. We begin by adding a static, uniform electric field:

$E \times B$ drift

Since plasma particles are charged, electric fields are present eg. through collective phenomena, external confining fields, single particle Coulomb interactions. The electric field component along B gives simple acceleration or deceleration. The electric field component perpendicular to B is more interesting. In Figure 2 one sees the trajectory in the perpendicular plane of an electron in crossed magnetic and electric fields.

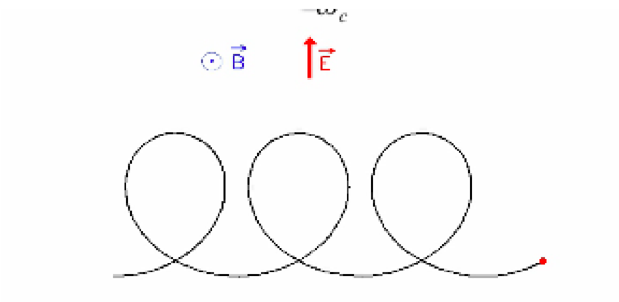

Figure 2: An electron in crossed $\vec{E}$ and $\vec{B}$ fields performs on average a sideways motion in the direction of $\vec{E} \times \vec{B}$.

The net effect is a motion in the $\vec{E} \times \vec{B}$ direction which has a steady state component. We now solve Newton's 2^{nd} law with an added electric field to find the expression for this sideways motion. Our equations now read:

$$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y$$

$$\frac{dv_y}{dt} = -\frac{qB_0}{m} v_x + \frac{q}{m} E_y = -\frac{qB_0}{m}(v_x - \frac{E_y}{B_0})$$

$$\frac{dv_z}{dt} = 0$$

E_y/B_0 is a velocity, we call it from now on v_E; it is constant since we assumed E and B constant.

$$\frac{dv_y}{dt} = -\frac{qB_0}{m}(v_x - v_E)$$

Since v_E is constant, we can substract it inside the d/dt of x-equation

$$\frac{d(v_x - v_E)}{dt} = \frac{qB_0}{m} v_y$$

Comparing with the original equations we had without an electric field makes us conclude that the new and the original equations have the same form, one only needs to replace v_x by v_x-v_E

$\frac{d(v_x - v_E)}{dt} = \frac{qB_0}{m} v_y$	$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y$
$\frac{dv_y}{dt} = -\frac{qB_0}{m}(v_x - v_E)$	$\frac{dv_y}{dt} = -\frac{qB_0}{m} v_x$
$\frac{dv_z}{dt} = 0$	$\frac{dv_z}{dt} = 0$
Now	**Before**

And we arrive at:

$v_x = \dfrac{E_y}{B_0} + v_\perp cos(\omega_c t + \delta)$	$v_x = v_\perp cos(\omega_c t + \delta)$
$v_y = -v_\perp sin(\omega_c t + \delta)$	$v_y = -v_\perp sin(\omega_c t + \delta)$
$v_z = v_\parallel$	$v_z = v_\parallel$
$\omega_c^2 = \left(\dfrac{qB_0}{m}\right)^2$	$\omega_c^2 = \left(\dfrac{qB_0}{m}\right)^2$
Now	**Before**

Here, we re-introduced E/B instead of v_E.
For situations with constant uniform E and B-fields, we can always define a local coordinate system where z is in the B-field direction and y is in the direction of the perpendicular component of E; hence, our derivation is valid in any coordinate system. The coordinate free formula for v_E is:

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2} = -\frac{\nabla\phi \times \vec{B}}{B^2}$$

The drift goes along constant electrostatic potential (ϕ) surfaces:

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2} = -\frac{\nabla\phi \times \vec{B}}{B^2}$$

Notice that the drift is independent of the particle! There is no reference to q or m in the formula. Thus, electrons and ions drift in the same direction, with the same speed.
It seems peculiar that the particle does not matter. However, this velocity can also be derived entirely without considering particles. If an observer moves at this velocity, the Lorenz transformation shows us that in this inertial frame (moving at the $E \times B$ velocity) there is no electric field:

$$E' = \gamma(\vec{E} + \vec{v} \times \vec{B}) + (1 - \gamma)\frac{\vec{v} \cdot \vec{E}}{v^2}\vec{v} =$$

$$\gamma\left(\vec{E} + \frac{\vec{E} \times \vec{B}}{B^2} \times \vec{B}\right) = \gamma(\vec{E} - \vec{E}) = 0$$

A charged particle therefore performs simple cyclotron motion in that frame (as long as $v_E = E/B < c$).

- Question: What happens when $E/B > c$?

$F \times B$ drift

The derivation we just performed only used Newton's 2^{nd} law. There was no reference made to the Lorenz transform or to Maxwell's equations. The Lorenz transform was only introduced after the result was derived - although it clearly could have been used instead, and one can argue that it would have been a more elegant derivation. However, the fact that only Newton's 2nd law was used is an advantage in the following, since the derivation can be rather trivially extended to cover other perpendicular forces. Recall the ExB drift derivation steps, and compare to a situation where a more general force **F** acts on the particle in a direction perpendicular t the magnetic field vector.

$$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y$$

$$\frac{dv_y}{dt} = -\frac{qB_0}{m} v_x + \frac{q}{m} E_y \longrightarrow -\frac{qB_0}{m} v_x + \frac{q}{m}\left(\frac{F_y}{q}\right)$$

$$\frac{dv_z}{dt} = 0$$

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2}$$

$$\vec{v}_F = \frac{\vec{F} \times \vec{B}}{qB^2}$$

As can be seen, the drift due to a general force generally is sensitive to the particle's charge.

Non-uniform B-field

Until now, we have assumed $\vec{B}$ to be straight and uniform. But in many types of magnetic traps for charged particles, the magnetic field is non-uniform. First, let us build some intuition on what happens with the particle orbits when the magnetic field strength is no longer spatially uniform. We assume here for simplicity that the B-field remains straight (in the $\hat{z}$ direction) but that the magnetic field strength varies in the $\hat{y}$ direction. It is clear from the expression for the Larmor radius, that the radius is larger in the region with lower B-field strength. Figure 3 shows a numerical simulation of a situation where this is the case. In order to show the effect clearly, the magnetic field changes substantially over a gyration. This breaks the assumptions that we will need in the following, but allows us to show in a couple of gyrations that the particle on average moves in the x-direction - perpendicular to ∇B and perpendicular to $\vec{B}$.
One can derive this drift by Taylor-expanding the B-field, taking advantage of the smallness of the Larmor radius (keeping only 1^{st} order terms):

$$B_z(x, y, z) = B_0 + \frac{\partial B_z}{\partial y}(y - y_{gc}) + O\left(\varepsilon^2\right)$$

$$\varepsilon = \left|\frac{\nabla B}{B}\right| r_L \ll 1$$

Such a derivation can be found in eg. [?] However, it is less algebra-intensive and more useful for the subsequent discussion to derive this drift velocity using the so-called first adiabatic invariant, μ. We therefore introduce the concept of adiabatic invariants now and then return to the derivation of the drift motion for this situation.

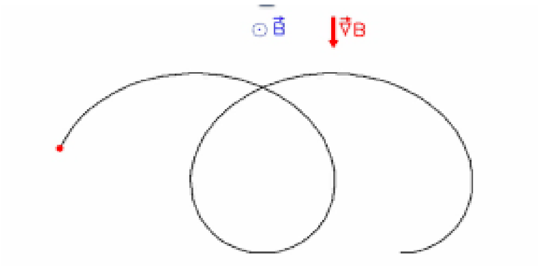

Figure 3: An electron orbit in a magnetic field whose strength varies significantly in a direction perpendicular to the direction of B itself (gradient pointing downward, ie. in the negative y-direction). The particle is seen to drift in the negative x-direction. One notices that the motion repeats itself shifted in the x-direction, indicating that the drift in the x-direction has a non-zero average value and will take the particle far

Adiabatic invariants

The concept of adiabatic invariants is known from analytic mechanics [**?**] and is only reviewed briefly here:
Assume that a particle performs periodic motion which can be described by a single coordinate q in a suitably defined spatial coordinate system. For example, a one-dimensional pendulum oscillating in the x-direction would have q=x. Then one can define the action as: $\oint p_q dq$. Here p_q is the generalized momentum associated with q.
If one perturbs the periodic motion by a small amount ε, the action remains conserved, to all powers in ε - one says that the action is *exponentially well conserved.*

We have already found one periodic motion for the charged particle in the magnetic field - the gyration. And the conditions for the guiding center approximation guarantee that the perturbation is small - the particle continues in an almost circular path since the magnetic field strength changes only a very small amount over the distance of a Larmor radius since $r_L << B/\nabla B$, and the magnetic field barely changes over the time scale of a single gyration, $\omega_c >> B/(\partial B/\partial t)$. The coordinate for the gyration is Θ, and $p_\Theta = mv_\Theta r$ is the associated generalized momentum (we recognize that this is just the angular momentum in the gyration). Thus, we can define the action for the gyration:

$$\oint p_q dq = \int_0^{2\pi} mv_\Theta r d\Theta = \int_0^{2\pi} mv_\perp r_L d\Theta = 2\pi mv_\perp r_L$$

Note that one performs the integral over the unperturbed orbit, ie. for a spatially and temporally constant magnetic field. Thus, the Larmor radius remains constant during the integral, at the previously calculated value. This can then be inserted into the formula for the action to yield:

$$\oint p_q dq = 2\pi mv_\perp \frac{mv_\perp}{qB} = \frac{4\pi m}{q} \frac{\frac{1}{2}mv_\perp^2}{B} = constant$$

Since π, m, and the charge q are fundamental constants, the following quantity μ is also constant:

$$\mu = \frac{\frac{1}{2}mv_\perp^2}{B} = constant$$

One uses this definition since it is more physically meaningful than the raw expression we derived for the action. With this definition, μB is equal to the perpendicular kinetic energy, and we now show that μ itself is the magnetic dipole moment of the charged particle, if we consider the particle as a charged current ring with radius r_L: This ring carries current $I = \frac{q\omega_c}{2\pi}$ since the electron gyrates and therefore carries a circular current. Thus, the magnetic moment $I \times A$ is:

$$IA = \frac{q\omega_c}{2\pi}\pi r_L^2 = \frac{q^2 B}{2\pi m}\pi\left(\frac{mv_\perp}{qB}\right)^2 = \frac{mv_\perp^2}{2B} = \mu$$

This magnetic dipole is anti-aligned with the magnetic field for both electrons and ions: Although electrons gyrate the other way around relative to ions, they have the opposite sign of charge, so the current and the magnetic field generated from it are in the same direction for electrons and ions. This shows that a plasma in a straight magnetic field is diamagnetic, it tends to weaken the magnetic field within which it is embedded.

A magnetic dipole with strength μ embedded in a magnetic field B anti-aligned to the dipole has potential energy μB, so it feels a force equal to $\vec{F} = -\mu\nabla B$. As a side remark, this force also works for neutral particles as long as they have a magnetic dipole moment. This is used to confine antihydrogen away from the walls.

If the $\mu\nabla B$ force is directed along B, it can provide some confinement along the field lines for charged particles - if they have sufficient μ and low enough parallel velocity, the force can turn them around and they reflect - it is as if an increasing magnetic field strength acts as a mirror that reflects the particles back. This so-called magnetic mirror confinement concept was pursued as a fusion energy confinement concept. However, this concept does not allow one to confine the entire distribution function, there will always be particles with low enough perpendicular kinetic energy (and therefore μ and high enough parallel kinetic energy that they are not reflected. This part of the distribution function then empties out of the trap - however the losses of plasma will continue as collisions continually repopulate this part of phase space.

If the force is perpendicular to B, then we get a drift. We use now the $F \times B$ formula we derived earlier:

$$\vec{F} = -\mu\nabla B = -\mu\frac{dB_z}{dy}\hat{y}$$

which we can also write as

$$\vec{v}_{\nabla B} = \frac{\vec{F}\times\vec{B}}{qB^2} = \frac{-\mu\nabla B\times\vec{B}}{qB^2} = \frac{mv_\perp^2}{2B}\frac{\vec{B}\times\nabla B}{qB^2}$$

Non-uniform B-field direction: Curvature drift

If the magnetic field strength is inhomogeneous, the magnetic field is usually also curved. Moreover, in order to confine charged particles with magnetic fields in a finite volume, the magnetic field lines can be curved around so they remain in a finite, closed volume. The curvature of the magnetic field leads to another drift:

In the guiding center approximation, the zeroth order motion of the particle (charged current ring) is the free-streaming along the magnetic field. So if the magnetic field is curved, the particle moves along this curved path and consequently feels a centrifugal force in the direction of the local radius of curvature of the magnetic field. This force then leads to a drift, which we can calculate using the FxB drift formula derived earlier. The centrifugal force is

$$\vec{F}_C = \frac{mv_\parallel^2}{R_C}\frac{\vec{R}_C}{R_C}$$

and therefore, the drift velocity is:

$$\vec{v}_{R_c} = \frac{\vec{F}_C \times \vec{B}}{qB^2} = \frac{mv_\parallel^2}{R_C^2}\frac{\vec{R}_C \times \vec{B}}{qB^2}$$

Combining the two drifts

One can show with a bit of algebra, that when there are no volumetric currents, and no strong time variation of the fields, Maxwell's equations imply that the gradient in the magnetic field strength and the curvature of the magnetic field are related one to one:

$$\left.\begin{array}{r}\nabla \times \vec{B} = \mu_0 \vec{j} = 0 \\ \nabla \cdot \vec{B} = 0\end{array}\right\} \Rightarrow \frac{\nabla B}{B} = -\frac{\vec{R}_C}{R_C^2}$$

Therefore we can combine the two non-uniformity drifts into one formula:

$$\vec{v}_{\nabla B + R_C} = \frac{2mv_\parallel^2 + mv_\perp^2}{2B}\frac{\vec{B} \times \nabla B}{qB^2}$$

One sees that the ∇B and curvature drifts cannot cancel or even offset each other. They act in the same direction. At this point, we should make sure that these drifts are small compared to the gyration velocity $v_\perp$ and the parallel velocity $v_{||}$. Again, we take a 100 eV electron in a magnetic field with B =1 T and $R_C = 1$ m:
$v_{Rc} \approx 100$ m/s vs $\approx 5 * 10^5$ m/s free streaming velocity and $r_L = 30\mu m << 1m$
This also serves to illustrate the need to include these drifts. Although they are negligible compared to the particle's thermal velocity, the particle will drift 1 m (the scale of the magnetic curvature) in 10 msec. Therefore, the deviation from the original field line on which it was gyrating is significant - the drift is important.

Mirror confinement

The mirror force $\mu\nabla B$ can also act along the magnetic field direction. This may seem counterintuitive since the Lorenz force has no component along $\vec{B}$.

There is no contradiction - the Lorenz force in a converging magnetic field region does create a force on a gyrating particle pushing it away from the region where the field lines are converging (ie. where the magnetic field strength is increasing). This is most easily illustrated for a situation where a particle is gyrating around a straight field line pointing in the z-direction, where the field strength is increasing in the z-direction. Applying a cylindrical coordinate system with the particle gyration center on the z-axis, an electron is gyrating in the positive θ direction, with $r = R_L$. An ion would be gyrating in the negative θ direction. $abla \cdot \vec{B} = 0$ then implies:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0 \tag{1}$$

where we can set $B_\theta = 0$ assuming for simplicity no volumetric currents. Then:

$$\frac{\partial}{\partial r}(rB_r) = -r\frac{\partial B_z}{\partial z} \Leftrightarrow rB_r = -\frac{r^2}{2}\frac{\partial B_z}{\partial z} \Leftrightarrow B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z} \tag{2}$$

where we have assumed that B_z does not vary significantly over a Larmor radius - since this is the distance over which we need to integrate, given that the particle is gyrating at $r = r_L$. Thus, for an increasing B_z, a negative B_r appears. Thus, there is a component of the Lorenz force that points in the (negative) z-direction. Note that for both negative and positive particles, the quantity $qv_\perp\hat{\theta}$ is negative since negative particles gyrate in the positive θ-direction and positive particles gyrate in the negative direction.

$$F_z\hat{z} = qv_\perp\hat{\theta} \times B_r\hat{r} = |q|v_\perp(-\frac{r_L}{2}\frac{\partial B_z}{\partial z}) = -\frac{mv_\perp^2}{2B}\frac{\partial B_z}{\partial z}\hat{z} \tag{3}$$

Note the vector identity $\hat{\theta} \times \hat{r} = -\hat{z}$ used here. The mirror force can be used to

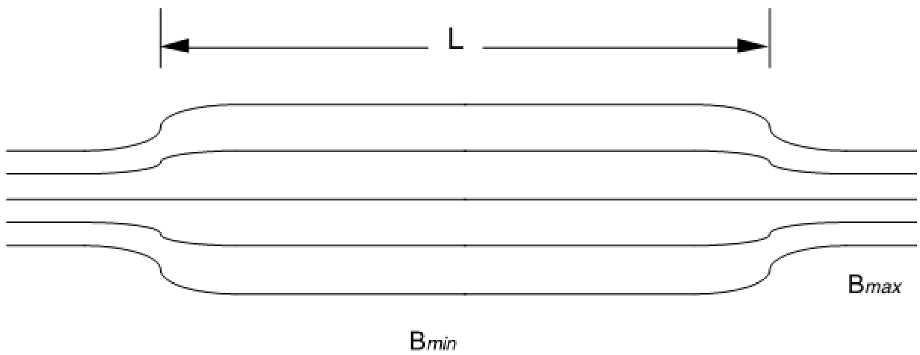

Figure 4: This figure shows the magnetic field lines in a simple cylindrical magnetic mirror device. Particles with sufficiently high μ and sufficiently low $v_{||}$ are trapped in the low field region and perform - in addition to the tiny gyrations with radius r_L - a periodic bounce motion in the parallel direction

confine guiding center particles in the third direction - along $\vec{B}$. This, it appears that we have identified a purely magnetic trap that confines particles. The classical magnetic mirror is a cylindrical magnetic topology, where at the two ends of the trap, the magnetic field increases from some low value B_0 to a larger value B_{max}, see Figure **??**. Let us investigate this confinement concept. For a static situation, the mechanical energy E of the charged particle is conserved. Assuming that the guiding center approximation is valid, μ is also conserved. Thus:

$$E = \frac{1}{2}mv_{\parallel}^2 + \frac{1}{2}mv_{\perp}^2 + q\phi = \frac{1}{2}mv_{\parallel}^2 + \mu B + q\phi \tag{4}$$

is conserved, and μB appears as a potential energy (the potential energy of a magnetic dipole in an external field). Setting $\phi = 0$ at this point (looking at pure magnetic confinement), the particle will be confined in a low-field region if and only if $E < \mu B_{max}$. A particle located in the low-B-field region ($B = B_0$) with perpendicular velocity $v_{\perp 0}$ and parallel velocity $v_{\parallel 0}$, has:

$$\mu = \frac{mv_{\perp 0}^2}{2B_0} \tag{5}$$

and

$$E = \frac{1}{2}mv_{\perp 0}^2 + \frac{1}{2}mv_{\parallel 0}^2 = \frac{1}{2}mv_0^2 \tag{6}$$

So the particle is confined if and only if

$$E < \mu B_{max} \Leftrightarrow \frac{1}{2}mv_0^2 < \frac{mv_{\perp 0}^2}{2B_0}B_{max} \Leftrightarrow \frac{B_0}{B_{max}} < \frac{v_{\perp 0}^2}{v_0^2} \Leftrightarrow \sqrt{\frac{B_0}{B_{max}}} < \sin(\theta) \tag{7}$$

In the last step we have introduced the so-called pitch angle θ - the angle between the charged particle's velocity vector and the magnetic field, $v_0 \sin(\theta) = v_{\perp 0}$. Thus, particles with a large pitch angle are confined, and particles with a small pitch angle are not. This means that a full Maxwellian distribution function cannot be confined - a Maxwellian includes all pitch angles, also the very small ones. The criterion in Eq. **??** can be rewritten in many forms, using the vector identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and the definitions $v_\perp = v\sin(\theta)$, $v_\parallel = v\cos(\theta)$.

Other adiabatic invariants

If the charged, magnetized particle performs another quasi-periodic motion in addition to the gyration, it will have another adiabatic invariant. The most common case is when the particle performs a periodic motion in its parallel motion. This can happen eg. when the particle is mirror trapped between two high field regions and bounces between them. The classical case here is the simple cylindrical magnetic mirror, sketched in Figure 4. This invariant is called the longitudinal invariant, usually noted J:

$$J = \oint v_\parallel dl$$

Another example is that of charged particles in a Penning trap. In this case, particles of one sign of charge are trapped electrostatically in the longitudinal direction by applied repulsive electrostatic potentials (Figure 5. In this case one can achieve full trapping at sufficiently large end bias voltages, but only for one sign of charge.

Time varying fields

0.1 Conservation of μ for time varying B-fields

Due to the aforementioned result from analytic mechanics, μ is conserved also when dB/dt is nonzero, as long as the B-field variation is small in one gyration and has no frequency component near ω_c. This implies that the perpendicular kinetic energy changes. One may wonder if this is really compatible with energy conservation, but the change in energy is due to the induced $\vec{E}$-field:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This can be shown explicitly relatively easily in simplified situations.

0.2 Time-varying electric fields - polarization drift

A time varying E-field gives rise to a so-called polarization drift (in addition to a time varying $E \times B$ drift):

$$\vec{v}_D = \frac{m^{\partial \vec{E}}/\partial t}{qB^2} \text{ if } \omega \ll \omega_c$$

The Derivation is most easily done doing a Fourier analysis. That is, assume that the time variation of the electric field takes a simple sinusoidal form:

$$\vec{E}(t) = \vec{E}_0 \cos(\omega t)$$

We then solve the equations of motion - this time taking into account that the term E/B is not time-independent. Although not necessary to solve the equations, the assumption $\omega << \omega_c$ makes the equations collapse into a form that is particularly compact, Eq. 0.2. For $\omega \approx \omega_c$ there is no μ conservation, and the polarization drift formula is no longer valid. The equations show a singularity for $\omega = \omega_c$ - this is the so-called cyclotron resonance, which one can be use for plasma heating, or for mass separation experiments (since ω_c depend on the mass of the charged particle).

Summary

In order to confine charged particles in a magnetic field for many single particle collisions the Larmor radius must be small. In this limit, full orbit calculations are very expensive, and generally not necessary. The guiding center approximation is very useful here:

- Charged gyrating particle is approximated as a charged current ring, with constant anti-aligned magnetic dipole moment, sliding along the magnetic field line in the ring center
- Such particles have slow drifts away from their "birth" magnetic field line, which can be calculated analytically

- One can calculate complicated trajectories rather straight-forwardly and accurately

The guiding center approximation can therefore be used to calculate whether orbits are confined or unconfined. This allows one to quickly identify configurations that are potentially good traps for charged particles, and in some cases, allows one to assess what the confinement time is in the presence of collisions. In the next lecture, we will analyze some toroidal magnetic configurations to determine if and how they confine charged particles.

Exercises