

# SIMONS SUMMER SCHOOL COMPUTER LAB PART I: ROTATING ELLIPSE EQUILIBRIUM CALCULATION

## 1. INTRODUCTION

In this activity we will use the VMEC equilibrium code to compute a vacuum magnetic field. This calculation will be performed at fixed-boundary, meaning that a boundary magnetic surface must be specified. We will choose this surface to be generated by a rotating ellipse geometry, discussed in section 2. We will compute the equilibrium as the parameters describing the geometry are varied and consider its effect on the rotational transform.

The VMEC code is widely used within the stellarator community. It solves the ideal MHD equilibrium equations,

$$\begin{aligned} (1a) \quad & \mathbf{J} \times \mathbf{B} = \nabla p(\psi) \\ (1b) \quad & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \\ (1c) \quad & \nabla \cdot \mathbf{B} = 0, \end{aligned}$$

where the pressure,  $p$ , is assumed to be a function of a flux surface label,  $\psi$ . Thus this calculation assumes the existence of nested, toroidal flux surfaces. The numerical method is based on the variational principle described in section 11.1 of the primer; an energy functional is minimized to obtain solutions to (1).

## 2. ROTATING ELLIPSE GEOMETRY

Throughout this activity we will represent position in  $\mathbb{R}^3$  using the standard cylindrical coordinates  $(R, \phi, Z)$  with  $\hat{\mathbf{R}} \times \hat{\phi} = \hat{\mathbf{Z}}$  (see figure 1). The position in a plane of constant  $\phi$  is defined by  $R \geq 0$  and  $Z$ .

Consider an ellipse in the  $R - Z$  plane centered at  $(R_0, Z_0)$  and parameterized by an angle  $\theta$ ,

$$\begin{aligned} (2a) \quad & R(\theta) = R_0 + a \cos(\theta) \\ (2b) \quad & Z(\theta) = Z_0 + b \sin(\theta), \end{aligned}$$

where  $a$  is the major axis of the ellipse and  $b$  is the minor axis. Thus the major axis is aligned with the  $\hat{\mathbf{R}}$  axis. If the  $a$  axis is rotated by an angle  $\alpha$  with respect to the  $\hat{\mathbf{R}}$  axis, the ellipse can be represented as,

$$\begin{aligned} (3) \quad & R(\theta) = R_0 + a \cos(\theta) \cos(\alpha) - b \sin(\theta) \sin(\alpha) \\ (4) \quad & Z(\theta) = Z_0 + b \sin(\theta) \cos(\alpha) + a \cos(\theta) \sin(\alpha). \end{aligned}$$

As an example, in figure 2a we plot an ellipse with  $R_0 = 5$ ,  $Z_0 = 0$ ,  $a = 2$ , and  $b = 1$  with  $\alpha = 0$  and  $\alpha = \pi/4$ .

We would like to consider a stellarator whose boundary magnetic surface is a rotating ellipse; thus the rotation of the ellipse,  $\alpha$ , should depend on the toroidal angle  $\phi$ . We consider the following

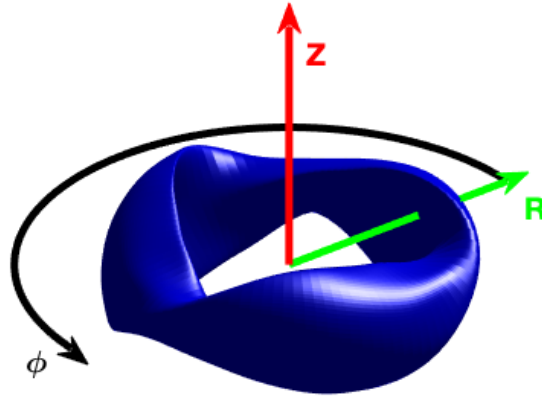
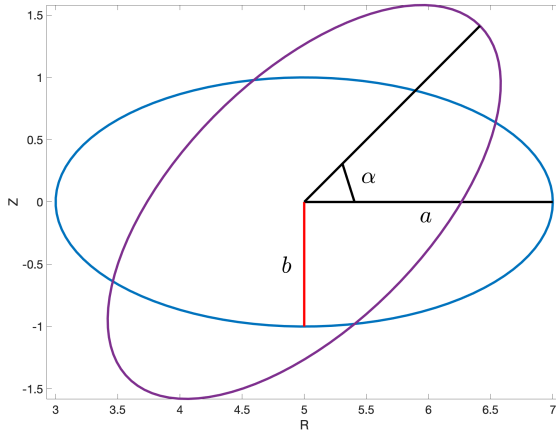
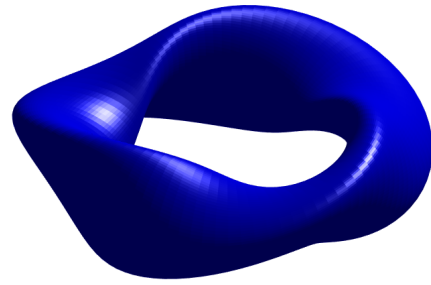


FIGURE 1. Canonical cylindrical coordinates



(A) A rotated ellipse



(B) A stellarator boundary surface generated by a rotated ellipse

FIGURE 2

representation,

$$(5) \quad R(\theta, \phi) = R_0 + a \cos(\theta) \cos\left(\frac{N_{FP}}{2}\phi\right) - b \sin(\theta) \sin\left(\frac{N_{FP}}{2}\phi\right)$$

$$(6) \quad Z(\theta, \phi) = Z_0 + b \sin(\theta) \cos\left(\frac{N_{FP}}{2}\phi\right) + a \cos(\theta) \sin\left(\frac{N_{FP}}{2}\phi\right),$$

where  $N_{FP}$ , the number of field periods, is an integer which counts the number of half rotations of the ellipse within one rotation of  $\phi$ . This representation ensures that the toroidal surface is periodic with respect to  $\phi$ .

An example is shown in figure 2b with  $R_0 = 5$ ,  $Z_0 = 0$ ,  $a = 2$ ,  $b = 1$ , and  $N_{FP} = 3$ .

### 3. VMEC REPRESENTATION OF BOUNDARY

For a fixed-boundary VMEC calculation, the shape of a bounding magnetic surface must be specified. Under the assumption of stellarator symmetry, the  $R$  and  $Z$  coordinates of the bounding surface can be represented in the following double Fourier series,

$$(7a) \quad R(\theta, \phi) = \sum_{m,n} \text{RBC}(\mathbf{n}, \mathbf{m}) \cos(m\theta - nN_{FP}\phi)$$

$$(7b) \quad Z(\theta, \phi) = \sum_{m,n} \text{ZBS}(\mathbf{n}, \mathbf{m}) \sin(m\theta - nN_{FP}\phi),$$

where  $\theta$  is any poloidal angle,  $\phi$  is the azimuthal angle of the cylindrical coordinate system, and  $N_{FP}$  is the number of periods.

We note that the angle,  $\theta$ , defined in the previous section is the angle with respect to the major axis of the ellipse. As the ellipse makes  $N_{FP}/2$  rotations during one toroidal turn, curves of constant  $\theta$  make  $N_{FP}/2$  rotations upon one toroidal turn. In order to define a poloidal angle, one which increases by  $2\pi$  upon a poloidal rotation without rotating toroidally, we will use  $\vartheta = \theta + \frac{N_{FP}}{2}\phi$ . In the  $(\vartheta, \phi)$  coordinate system our surface is represented as

$$(8) \quad R(\vartheta, \phi) = R_0 + a \cos\left(\vartheta - \frac{N_{FP}}{2}\phi\right) \cos\left(\frac{N_{FP}}{2}\phi\right) - b \sin\left(\vartheta - \frac{N_{FP}}{2}\phi\right) \sin\left(\frac{N_{FP}}{2}\phi\right)$$

$$(9) \quad Z(\vartheta, \phi) = Z_0 + b \sin\left(\vartheta - \frac{N_{FP}}{2}\phi\right) \cos\left(\frac{N_{FP}}{2}\phi\right) + a \cos\left(\vartheta - \frac{N_{FP}}{2}\phi\right) \sin\left(\frac{N_{FP}}{2}\phi\right).$$

To represent the rotating ellipse boundary in the double Fourier series, we need to apply several trig identities to obtain,

$$(10a) \quad \begin{aligned} R(\vartheta, \phi) &= R_0 + 0.5a (\cos(\vartheta - N_{FP}\phi) + \cos(\vartheta)) \\ &\quad - 0.5b (\cos(\vartheta - N_{FP}\phi) - \cos(\vartheta)) \end{aligned}$$

$$(10b) \quad \begin{aligned} Z(\vartheta, \phi) &= Z_0 + 0.5b (\sin(\vartheta) + \sin(\vartheta - N_{FP}\phi)) \\ &\quad + 0.5a (\sin(\vartheta) - \sin(\vartheta - N_{FP}\phi)). \end{aligned}$$

We will use this boundary to compute the vacuum magnetic field using the VMEC code. In the following section we review the pertinent input parameters.

### 4. VMEC INPUT PARAMETERS

- **LASYM**: a Boolean determining whether stellarator asymmetric modes should be included in the equilibrium calculation. Stellarator symmetry is implied by the rotating ellipse boundary, as from (10) we note that  $R(\theta, \phi)$  is described by only cosine modes, while  $Z(\theta, \phi)$  is described by only sine modes. Therefore, **LASYM** should be set to **F** for these calculations.
- **RBC**( $\mathbf{n}, \mathbf{m}$ ) and **ZBS**( $\mathbf{n}, \mathbf{m}$ ): the Fourier coefficients describing the boundary surface according to (7). Here  $n$  is a toroidal mode number and  $m$  is a poloidal mode number. Note that VMEC will return with an error if **RBC**(0,1) is zero. Even if it is not needed to describe the desired boundary surface, it should be set to a very small, non-zero value.
- **NFP**: the number of field periods. This should be consistent with the representation in (7).

- **MPOL** and **NTOR**: the number of poloidal and toroidal mode numbers used to compute the equilibrium solution with a Fourier discretization. The solution will be discretized for  $m \in [0, \text{MPOL} - 1]$  and  $n \in [-\text{NTOR}, \text{NTOR}]$ .
- **PHIEDGE**: the value of the toroidal flux in SI units through the boundary magnetic surface. This sets the overall scale of the equilibrium magnetic field. See the definition of the toroidal flux in section 6.3 of the primer.
- **NCURR**: an integer determining whether the equilibrium calculation is performed at fixed rotational transform (**NCURR**=0) or at fixed toroidal current (**NCURR**=1). We will be using **NCURR**=1 in order to enforce a vacuum solution. In this case the toroidal current must be specified by the **AC** coefficients.
- **AC**: a polynomial description of the integrated toroidal current with respect to  $s = \Phi/\Phi_{\text{edge}}$ , where  $\Phi$  is the toroidal flux,

$$I_T'(s) = \sum_i \text{AC}(i) s^{i-1}.$$

Note that this is represented as a power series for the derivative of  $I_T(s)$  by default. For a vacuum solution these coefficients should be set to zero. See section 9.2 for the primer for a definition of  $I_T$ .

- **AM**: a polynomial description of the pressure with respect to  $s$ ,

$$p(s) = \sum_i \text{AM}(i) s^{i-1}.$$

For a vacuum field solution, these coefficients should also be set to zero.

- **RAXIS** and **ZAXIS**: Fourier modes describing the initial guess for the magnetic axis curve,  $(R_a(\phi), Z_a(\phi))$ ,

$$R_a(\phi) = \sum_n \text{RAXIS}(n) \cos(-nN_{FP}\phi)$$

$$Z_a(\phi) = \sum_n \text{ZAXIS}(n) \sin(-nN_{FP}\phi).$$

For the rotating ellipse representation, a good initial guess is **RAXIS**(0) =  $R_0$  and **ZAXIS**(0) =  $Z_0$ .

- **NS\_ARRAY**: an array of the number of radial gridpoints to use during each iteration of the calculation. Each element defines the number of magnetic surfaces to include in the calculation at each iteration. The VMEC solution is computed using a multigrid method, beginning with a small number of surfaces and increasing until the desired resolution is achieved. In order to achieve convergence, it is typically necessary to begin with a small number of surfaces (10-20) and increase to your desired resolution (typically 75-150 is sufficient) in increments of 20-40.
- **FTOL\_ARRAY**: an array defining the tolerances in the force residual used at each grid level. This should have the same number of elements as **NS\_ARRAY**. Typically the finest grid should have a value of  $10^{-11} - 10^{-15}$ . The coarse grids can have larger tolerances. The VMEC calculation is performed by minimizing an energy functional until this normalized tolerance in the force residual is achieved.
- **NITER**: The maximum number of iterations to use at each iteration of the calculation. This array should be of the same size as **FTOL\_ARRAY** and **NS\_ARRAY**. If the number of iterations exceeds **NITER** during the finest grid evaluation, the code will exit with an error. If it

exceeds NITER during the coarser grid evaluations, the calculation will proceed to the next grid size defined by the next element of NS\_ARRAY.

- NSTEP: the number of iterations between output of the force residual as the energy is minimized.
- DELT: this parameter controls the stepsize in the minimization of the energy functional. Typically values in the range 0.2-0.9 are sufficient. This control parameter should not be changed unless one is having difficulty obtaining convergence.

For further documentation, see the [VMEC wiki](#).

## 5. SUGGESTED ACTIVITIES

VMEC should be run from the Macs in the computer or the PPPL cluster by following the instructions outlined in the ‘Simons summer school computer lab’ document. The output can be plotted with either the provided MATLAB script, `plot_vmec.m`, or python script, `vmecPlot`.

- (1) Run the example input file given in the `vmec_inputs/rotating_ellipse` directory. This example corresponds to  $R_0 = 0$ ,  $Z_0 = 0$ ,  $a = 2$ ,  $b = 1$ , and  $N_{FP} = 3$ . This will create an output file in NETCDF format, `wout_rotating_ellipse.nc`.
- (2) Plot the equilibrium solution using either the python or MATLAB scripts. This will generate the profile of the rotational transform as a function of  $s$  and the shapes of the magnetic surfaces at several cross-sections.
- (3) Try playing with the parameters describing the rotating ellipse representation ( $R_0$ ,  $Z_0$ ,  $N_{FP}$ ,  $a$ ,  $b$ ) by adjusting the corresponding VMEC parameters (RAXIS, ZAXIS, RBC, ZBS, NFP). Check that the cross-sections of the boundary surface match with your desired geometry.
- (4) Note how the magnitude of the rotational transform changes as you adjust these parameters. What do you expect to see based on the discussion in sections 7.6 and 10.4 of the primer?
- (5) *Challenge activity* Try adding plasma pressure and toroidal current by increasing the values of the AM and AC coefficients. Use a linear approximation for  $p(s)$  and  $I_T(s)$  that vanishes at  $s = 1$ ; thus  $AC(i > 1) = 0$ ,  $AM(1) = -AM(2)$ , and  $AM(i > 2) = 0$ . How does this change the shapes of the interior surfaces (this can be seen with the python script)?
- (6) For those interested, we have included input files for the W7-X and NCSX stellarator boundaries in the `vmec_inputs/W7X` and `vmec_inputs/NCSX` directories. Run these examples and plot the results with the MATLAB or python script. How do these configurations differ from the rotating ellipse configurations? Can you approximate any of the interior surfaces by a rotating ellipse boundary?