Exercises for the guiding center approximation and adiabatic invariants

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Problem 1

Single particle orbit in the low magnetic field limit.

In class, we found that charged particles drift across the magnetic field at the ExB drift velocity. In the inertial frame moving at this velocity, the electric field vanishes. This is valid for E/B<c. Imagine what happens in the opposite case, E/B>c. Note that this occurs when B is small.

a) If B is in the z direction and E is in the x direction, and you start a charged, massive particle at zero velocity at (x,y)=(0,0), will the particle ever come back to the x=0 plane?
b) Find an inertial frame in which the particle experiences a simple linear acceleration. In order to do this, you need to consider the Lorentz transformation of the electric and magnetic fields from a stationary frame to a frame moving at velocity v. Note that when E/B<c, the frame moving at the ExB velocity is precisely the frame in which the electric field is zero. However, in the situation you now need to consider, the ExB velocity is larger than the speed of light so you cannot transform to the ExB frame anymore.

The electric and magnetic fields E' and B' in a frame moving at velocity v are related to the electric and magnetic fields E and B in a stationary frame by the Lorentz transformation (SI units):

$$\vec{E}' = \gamma\left(\vec{E} + \vec{v} \times \vec{B}\right) - \frac{\gamma^2}{\gamma + 1}\frac{\vec{v}}{c}\left(\frac{\vec{v}}{c} \cdot \vec{E}\right)$$

$$\vec{B}' = \gamma\left(\vec{B} - \frac{\vec{v}}{c^2} \times \vec{E}\right) - \frac{\gamma^2}{\gamma+1}\frac{\vec{v}}{c}\left(\frac{\vec{v}}{c} \cdot \vec{B}\right)$$

Hint: Note that there is some symmetry between E and B in the Lorentz transform. You may be able to guess what frame of reference eliminates the B-field in this case, by considering this symmetry.

Useful information: the classical expression for the Lorentz force is relativistically correct, ie. it is correct also for particle velocities approaching the speed of light.

Problem 2

An electron (charge –e, mass m_e) is in a cylindrically symmetric mirror trap with an applied, uniform electric field. Use cylindrical coordinates (r,θ,z). The particle has its guiding center located at r=0. On the r=0 line, the electric and magnetic fields are both time-independent and point in the z-direction and have the following magnitudes:

$E_z=E_0$, $B_z=B_0(1+z^2/(z_0)^2)$.

Here E_0, B_0 and z_0 are positive constants. The electron has $v_\perp= v_0$ and $v_\parallel=0$ and is located at z=0. You may assume that v_0 is nonzero but small enough that the Larmor radius is small compared to all other spatial scales in the problem.

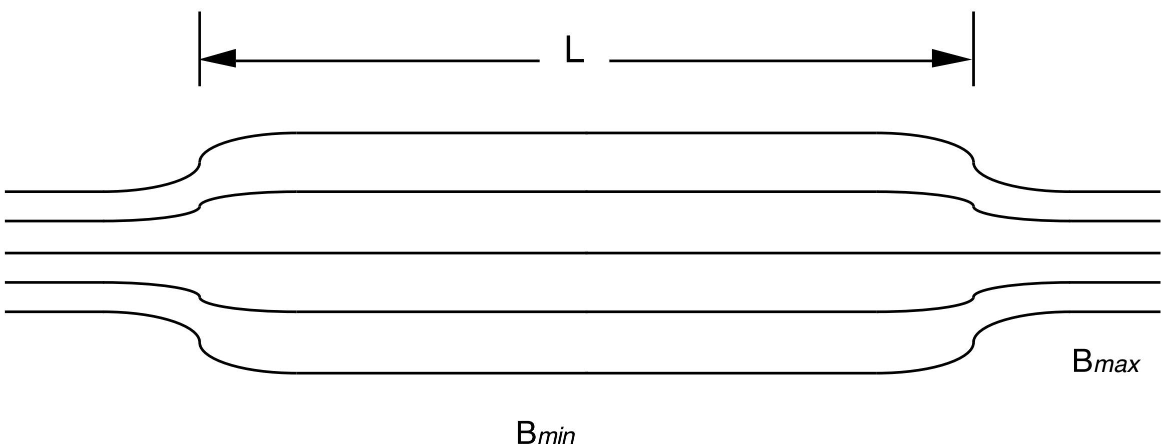
a) Show that the particle is going to perform sinusoidal oscillations in the z direction, and express the oscillation frequency in terms of the parameters given in the problem.

b) What is the oscillation amplitude for the electron's motion?

c) Find the z-location at which an electron with perpendicular velocity v_0 and zero parallel velocity, and its guiding center at r=0, will remain at rest at that z-location – ie. find the equilibrium z-location for such an electron.

Problem 3

Consider a charged particle trapped in a cylindrically symmetric magnetic mirror system with highly localized mirror regions where the magnetic field changes from B_{min} to B_{max} with $B_{max}=3\ B_{min}$ over a distance << L:



(a) If a proton in the region where $B=B_{min}$ has $v_\perp = 2v_{||} = 2v_o$, compute the value of the magnetic moment, μ, and estimate the second adiabatic invariant, $J = \int v_{||}\ dl$

(b) If we slowly increase the magnetic field preserving the ratio of B_{max}/B_{min}, will the proton escape from the mirror system? If so, by what factor must the field be increased to allow the proton to escape from the mirror?