Magnetic confinement of charged particles

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Overview

- Motivation for studying single particle orbits in magnetic fields
- Motion in a straight uniform B-field (basic review)
- The guiding center approximation
- Motion including a static electric field, or gravity
- Motion in inhomogeneous and bent B-fields and the first adiabatic invariant
- Mirror confinement
- Time variation (no derivation)

The need for magnetic field confinement

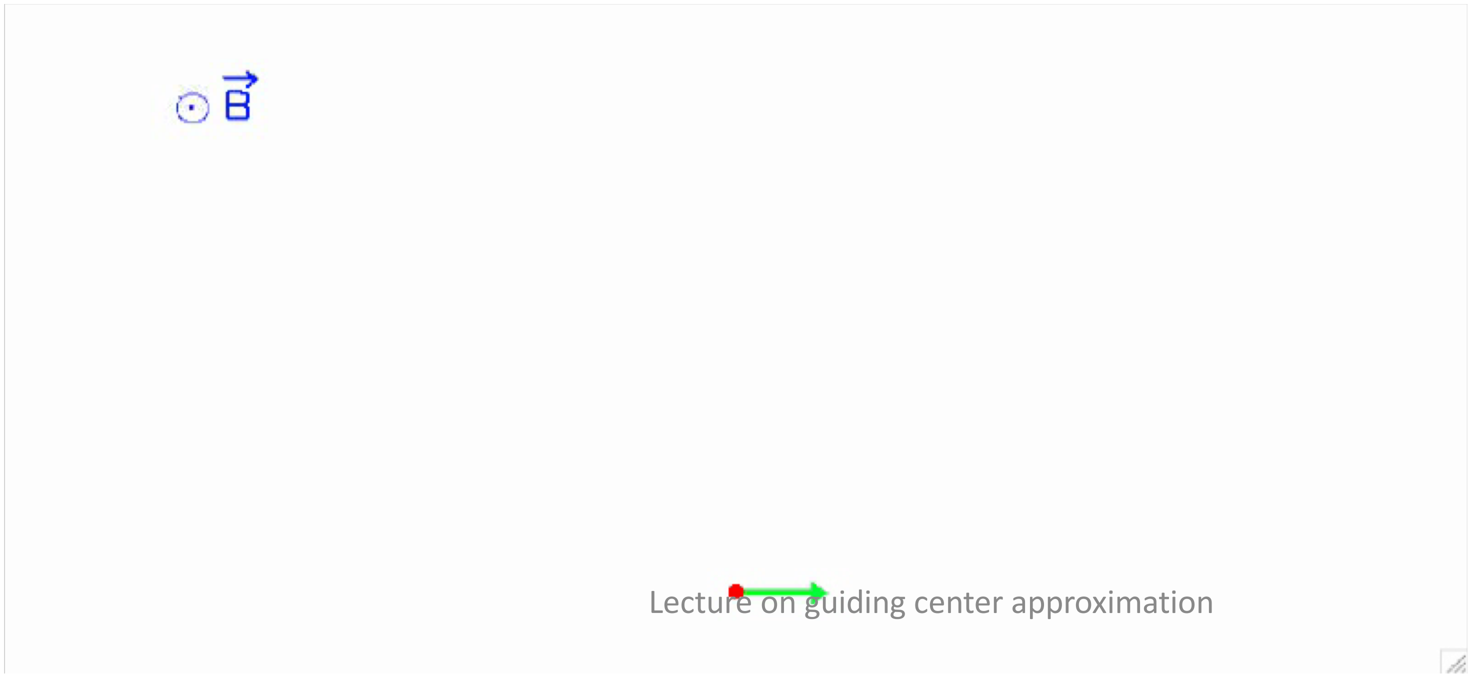
- The easiest fusion process to reach is D-T fusion
- This requires particle kinetic energies in the range 10-100 keV
- Even at the particle energy of peak D-T fusion reactivity, non-fusion collisions (scattering) dominate over the fusion collisions by two orders of magnitude
  - Must confine plasma at T>10 keV (~120 M Kelvin) for many collisions
- Thermal speed of D and T is on the order of 10^6 m/s at these temperatures (and even higher for electrons) – µs confinement if you have no confining field?
- Electric fields alone won't work: Confine only one species
- Magnetic fields may work (must be bent!)
- Gravity works on the sun, but not on Earth

Charged particle motion in a straight magnetic field

- A charged particle performs a screw-like path if it is confined by a straight uniform magnetic field and it feels no other forces
- Start with Newton's 2nd law and the Lorentz force:

$$\vec{F} = m\vec{a} = q\vec{v} \times \vec{B}$$

$$\vec{B} = B_0\hat{z}$$



Charged particle motion in a straight magnetic field

- Newton's 2nd law written coordinate by coordinate:

$$\vec{F} = m\vec{a} = q\vec{v} \times \vec{B} \Rightarrow$$

$$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y$$

$$\frac{dv_y}{dt} = -\frac{qB_0}{m} v_x$$

$$\frac{dv_z}{dt} = 0$$

Charged particle motion in a straight magnetic field

- Newton's 2nd law written coordinate by coordinate:
- qB_0/m is an inverse time scale

$$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y$$

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$$\frac{dv_z}{dt} = 0$$

Charged particle motion in a straight magnetic field

- qB_0/m is an inverse time scale: give it it's own symbol

$$\frac{dv_x}{dt} = \omega_c v_y$$

$$\frac{dv_y}{dt} = -\omega_c v_x$$

$$\frac{dv_z}{dt} = 0$$

$$\omega_c = \frac{qB_0}{m} \quad \text{- sometimes } \omega_c = \frac{|q|B_0}{m}$$

Charged particle motion in a straight magnetic field

- Decouple v_x and v_y equations:

$$\frac{d}{dt}\left[\frac{dv_x}{dt} = \omega_c v_y\right]$$

$$\frac{dv_y}{dt} = -\omega_c v_x$$

$$\frac{dv_z}{dt} = 0$$

$$\omega_c = \frac{qB_0}{m} \text{ - sometimes } \omega_c = \frac{|q|B_0}{m}$$

Charged particle motion in a straight magnetic field

- Eliminate v_y from v_x equation by differentiation and substitution
- V_z equation is trivial

$$\left.\begin{aligned} \frac{d^2 v_x}{dt^2} &= \omega_c \frac{dv_y}{dt} \\ \frac{dv_y}{dt} &= -\omega_c v_x \end{aligned}\right\} \Rightarrow \frac{d^2 v_x}{dt^2} = -\omega_c^2 v_x$$

$$\frac{dv_z}{dt} = 0 \Rightarrow v_z = \text{constant} = \mathrm{v}_{||}$$

$$\omega_c^2 = \left(\frac{qB_0}{m}\right)^2$$

Charged particle motion in a straight magnetic field

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- We recognize the simple harmonic oscillator for v_x
- Find v_y by differentiation
- V_z equation is trivial:

$$\frac{dv_z}{dt} = 0 \Rightarrow v_z = \text{constant} = v_{\parallel}$$

$$\omega_c^2 = \left(\frac{qB_0}{m}\right)^2$$

Charged particle motion in a straight magnetic field

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$$\left.\begin{aligned} \frac{dv_x}{dt} &= \omega_c v_y \\ \frac{dv_y}{dt} &= -\omega_c v_x \end{aligned}\right\} \Rightarrow \begin{aligned} \frac{d^2 v_x}{dt^2} &= -\omega_c^2 v_x \\ v_y &= \frac{1}{\omega_c}\frac{dv_x}{dt} \end{aligned}$$

- We recognize the simple harmonic oscillator for v_x
- Find v_y by differentiation
- V_z equation is trivial:

$$\frac{dv_z}{dt} = 0 \Rightarrow v_z = \text{constant} = v_{\parallel}$$

$$\omega_c^2 = \left(\frac{qB_0}{m}\right)^2$$

Charged particle motion in a straight magnetic field

- Velocity components:

$$v_x = v_\perp \cos(\omega_c t + \delta)$$

$$v_y = -v_\perp \sin(\omega_c t + \delta)$$

$$v_z = \mathrm{v}_{\parallel}$$

$$\omega_c^2 = \left(\frac{qB_0}{m}\right)^2$$

- Next step: integrate to get position

Charged particle motion in a straight magnetic field

- Integrate in time to get position
- Define Larmor radius and guiding center:

$$v_x = v_\perp \cos(\omega_c t + \delta) \Rightarrow x = x_{gc} + \frac{v_\perp}{\omega_c}\sin(\omega_c t + \delta)$$

$$v_y = -v_\perp \sin(\omega_c t + \delta) \Rightarrow y = y_{gc} + \frac{v_\perp}{\omega_c}\cos(\omega_c t + \delta)$$

$$v_z = \mathrm{v}_\parallel \Rightarrow z = z_0 + \mathrm{v}_\parallel t$$

$$\omega_c = \frac{qB_0}{m} \text{ (cyclotron frequency), } \mathrm{f_c} = \omega_c / 2\pi$$

$$\mathrm{r_L} = \left|\frac{v_\perp}{\omega_c}\right| = \left|\frac{mv_\perp}{qB_0}\right| \text{ (Larmor radius, gyroradius)}$$

The guiding center approximation:

- If the Larmor radius is on the order of the size of the confining field, then only collisionless orbits are confined:
    - After a collision, the particle will have a new guiding center, about one Larmor radius away from the original guiding center
    - If this new Larmor orbit intersects material walls or extends to regions of much lower B-field, the particle is not confined
- In order to confine charged particles magnetically for many collision times, the Larmor radius must be small!
- When the Larmor radius is small, and the cyclotron frequency is large:
- one can derive analytic formulas for the time evolution of the guiding center (x_{gc}, y_{gc}, z), averaging over the gyration

$$r_L << B / \nabla B \text{ and } \omega_c >> \frac{\partial B}{\partial t} \frac{1}{B} \quad (*)$$

The guiding center approximation is often necessary:

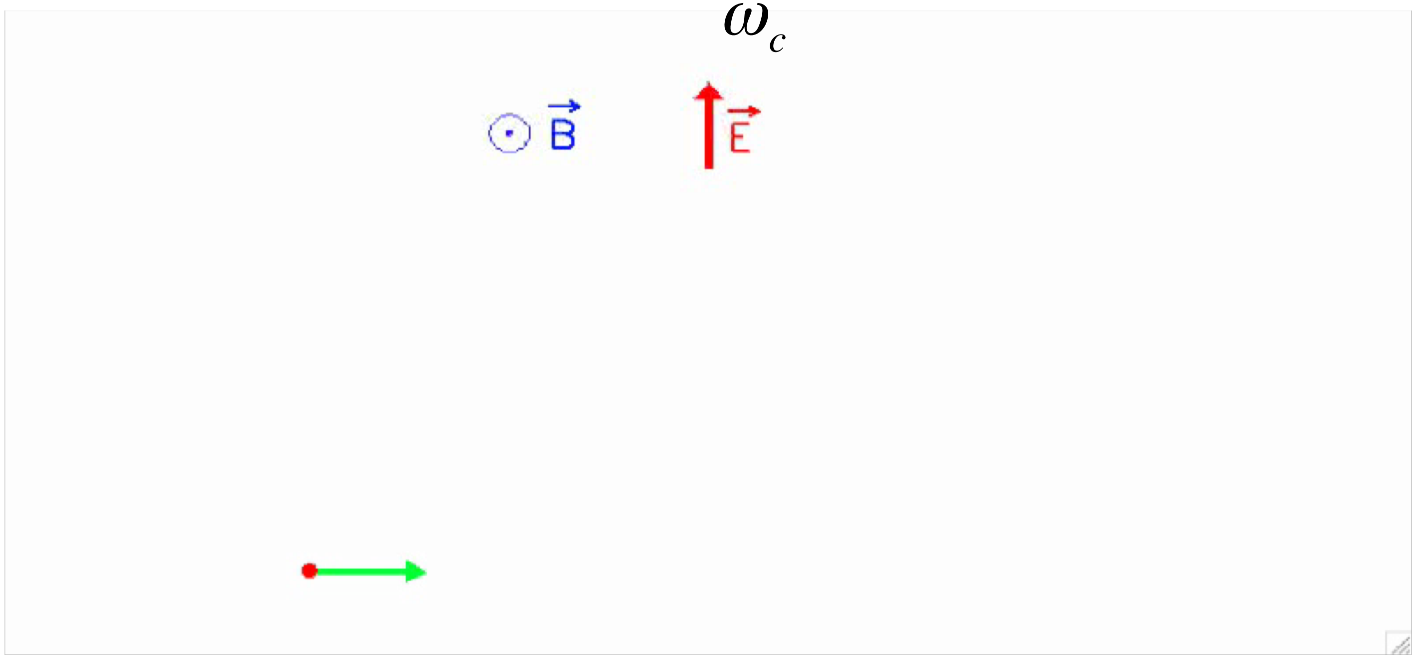
- For a 1 eV electron in a B=1 T field, the cyclotron frequency is large and the Larmor radius small:

$$\omega_c \approx 1.8 \times 10^{11} s^{-1}$$
$$r_L \approx 3 \times 10^{-6} m$$

- Just to follow the electron for one microsecond requires $>10^6$ time steps if a simple numerical scheme is used.
- Almost the full computational effort is spent calculating the circular motion….
- Averaging over the gyromotion allows fast and accurate calculations of the motion of charged particles in a magnetic field, both analytic and numerical
- Essentially, the gyrating particle is replaced by a charged (q), massive (m) ring of current ($I=e\omega_c/2\pi$) , with its center at the particle's gyrocenter.
- We will do this for a few important cases in the following:

ExB drift

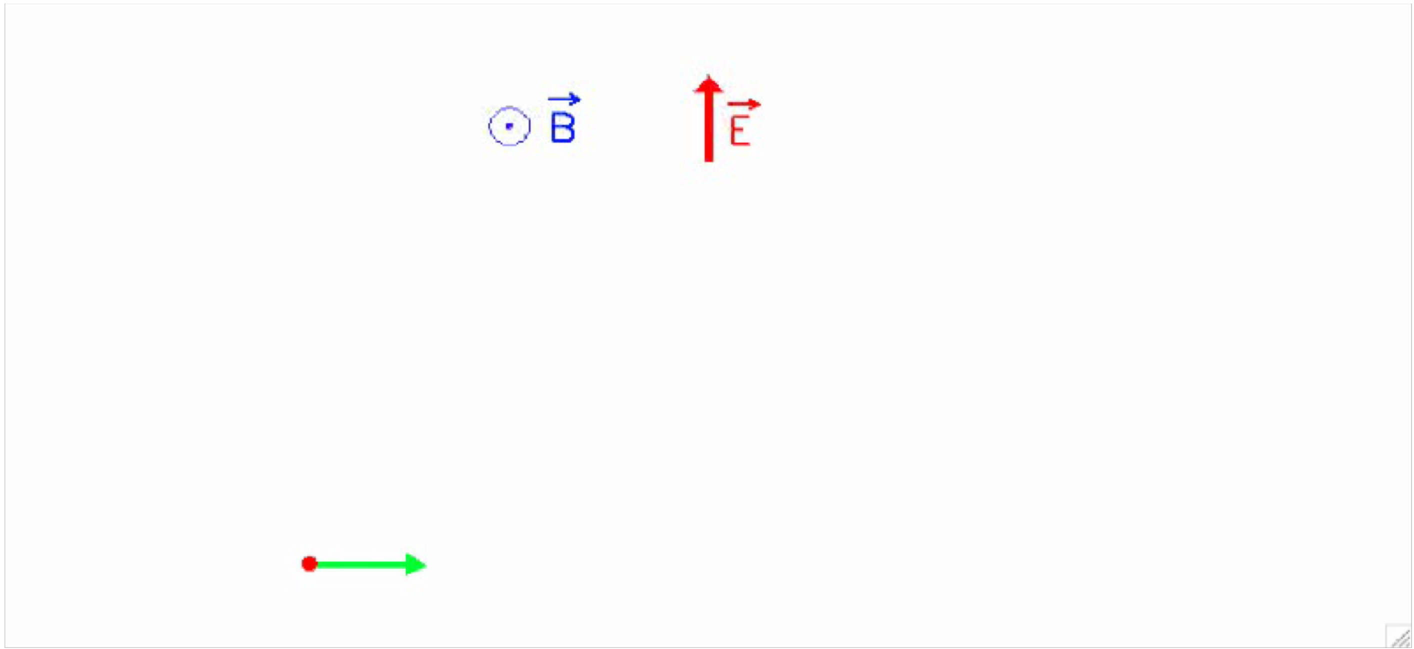
- Since particles are charged, the electric field naturally enters the equations (collective phenomena, external confining fields, single particle Coulomb interactions)
- Electric field component along B gives simple acceleration or deceleration
- Electric field component perpendicular to B is more interesting:
- Example below: Electron $r_L = \dfrac{v_\perp}{\omega_c}$



ExB drift

- Net effect is a motion in the ExB direction which has a steady state component

  - Let’s calculate this drift:



ExB drift

- Newton's 2nd law written coordinate by coordinate (again)
- Let's add an electric field now

$$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y$$

$$\frac{dv_y}{dt} = -\frac{qB_0}{m} v_x$$

$$\frac{dv_z}{dt} = 0$$

ExB drift

$$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y$$

$$\frac{dv_y}{dt} = -\frac{qB_0}{m} v_x + \frac{q}{m} E_y$$

$$\frac{dv_z}{dt} = 0$$

ExB drift

- E_y/B_0 is a velocity, v_E; it is constant since we assumed E and B constant.

$$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y$$

$$\frac{dv_y}{dt} = -\frac{qB_0}{m}(v_x - \frac{E_y}{B_0})$$

$$\frac{dv_z}{dt} = 0$$

ExB drift

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$$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y$$

$$\frac{dv_y}{dt} = -\frac{qB_0}{m}(v_x - v_E)$$

$$\frac{dv_z}{dt} = 0$$

ExB drift

- E/B is a velocity, v_E; it is constant since we assumed E and B constant.
- Since v_E is constant, we can subtract it inside the d/dt of x-equation

$$\frac{d(v_x - v_E)}{dt} = \frac{qB_0}{m} v_y$$

$$\frac{dv_y}{dt} = -\frac{qB_0}{m}(v_x - v_E)$$

$$\frac{dv_z}{dt} = 0$$

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Before

ExB drift

- It's clear we have the same equation as before, just replacing v_x by v_x-v_E

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Before

ExB drift

- It's clear we have the same equation as before, just replacing v_x by v_x-v_E

$$v_x - v_E = v_\perp \cos(\omega_c t + \delta)$$
$$v_y = -v_\perp \sin(\omega_c t + \delta)$$
$$v_z = \mathrm{v}_{||}$$
$$\omega_c^2 = (\frac{qB_0}{m})^2$$

Now

$$v_x = v_\perp \cos(\omega_c t + \delta)$$
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Before

ExB drift

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$$v_x = \frac{E_y}{B_0} + v_\perp \cos(\omega_c t + \delta)$$

$$v_y = -v_\perp \sin(\omega_c t + \delta)$$

$$v_z = \mathrm{v}_{\parallel}$$

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Before

ExB drift: Coordinate-free formulation

•For situations with constant uniform E and B fields, we can always define a local coordinate system where z is in the B-field direction and y is in the direction of the component of E perpendicular to B; hence, our derivation is valid in any coordinate system. The coordinate free formula for v_E is:

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2} = -\frac{\nabla\phi \times \vec{B}}{B^2}$$

•The drift is independent of the particle! No reference to *q* or *m*

•Same for ions and electrons

•The drift goes along constant ϕ surfaces – does not change the electrostatic energy of the particle

ExB drift: Coordinate-free derivation

- We can also prove that the particles ExB drift without using coordinates

$$\vec{v}_E = \frac{\vec{E}\times\vec{B}}{B^2} = -\frac{\nabla\phi\times\vec{B}}{B^2}$$

- We know what we are looking for: $\vec{\mathrm{v}}_\perp = \vec{v}_g + \vec{v}_E$

$$m\frac{d\vec{v}}{dt} = q\left(\overrightarrow{E_\perp} + \vec{v}\times\vec{B}\right) \Rightarrow m\frac{d\vec{v}_\perp}{dt} = q\left(\overrightarrow{E_\perp} + \ \vec{v}_E\times\vec{B} + \vec{v}_g\right)$$

$$= q\left(\overrightarrow{E_\perp} + \left(\vec{E}\times\vec{B}\right)\times\frac{\vec{B}}{B^2} + \vec{v}_g\right) = q\vec{v}_g\times\vec{B} \Leftrightarrow m\frac{d\vec{v}_g}{dt} = q\vec{v}_g\times\vec{B}$$

ExB drift

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2} = -\frac{\nabla\phi \times \vec{B}}{B^2}$$

•The drift is independent of the particle! There is no reference to *q* or *m*

•Same for ions and electrons, pions, all charged particles….why?

Answer: In the inertial frame that moves at the ExB velocity, there is no E-field!

$$E' = \gamma(\vec{E} + \vec{v} \times \vec{B}) + (1-\gamma)\frac{\vec{v}\cdot\vec{E}}{v^2}\vec{v} =$$

Lorentz transform:

$$\gamma(\vec{E} + \frac{\vec{E} \times \vec{B}}{B^2} \times \vec{B}) = \gamma(\vec{E} - \vec{E}) = 0$$

•A charged particle therefore performs simple cyclotron motion in that frame (as long as v_E=E/B<c)

•Exercise: What happens when E/B>c?

FxB drift

- The derivation we did only used Newton's 2nd law – no reference to the Lorentz transform or Maxwell's equations
    - (then afterwards it was realized that we could have used Lorentz)
- But this is actually an advantage: Our derivation can be trivially extended to any other constant perpendicular force acting on our particle:

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2}$$

$$\vec{v}_F = \frac{\vec{F} \times \vec{B}}{qB^2}$$

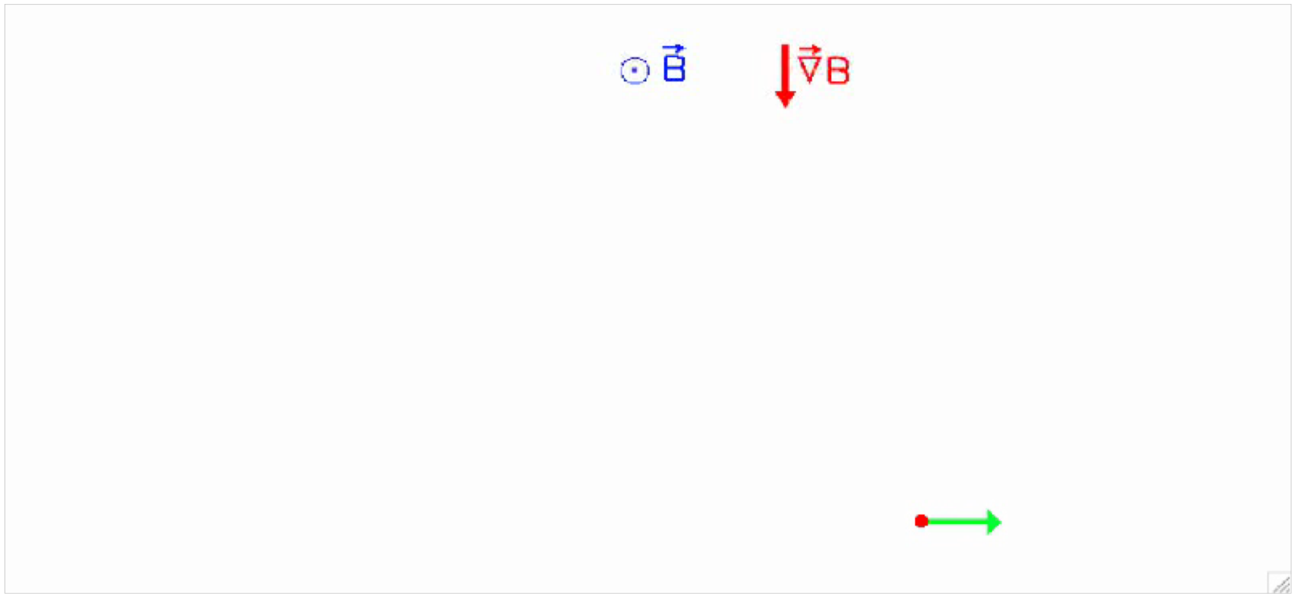
- The general force drift cares about the particle's charge, as one would expect. Example F=mg leads to a gravitational drift in opposite directions for electrons and ions.

Non-uniform B-field

- In some traps, the magnetic field is non-uniform
- We assumed B straight and uniform
- What happens when (for example) the B-field strength changes spatially?

Non-uniform B-field

- In some traps, the magnetic field is non-uniform
- We assumed B straight and uniform
- What happens when (for example) the B-field strength changes spatially?
- Assume for simplicity here: B-field is straight but increases in strength

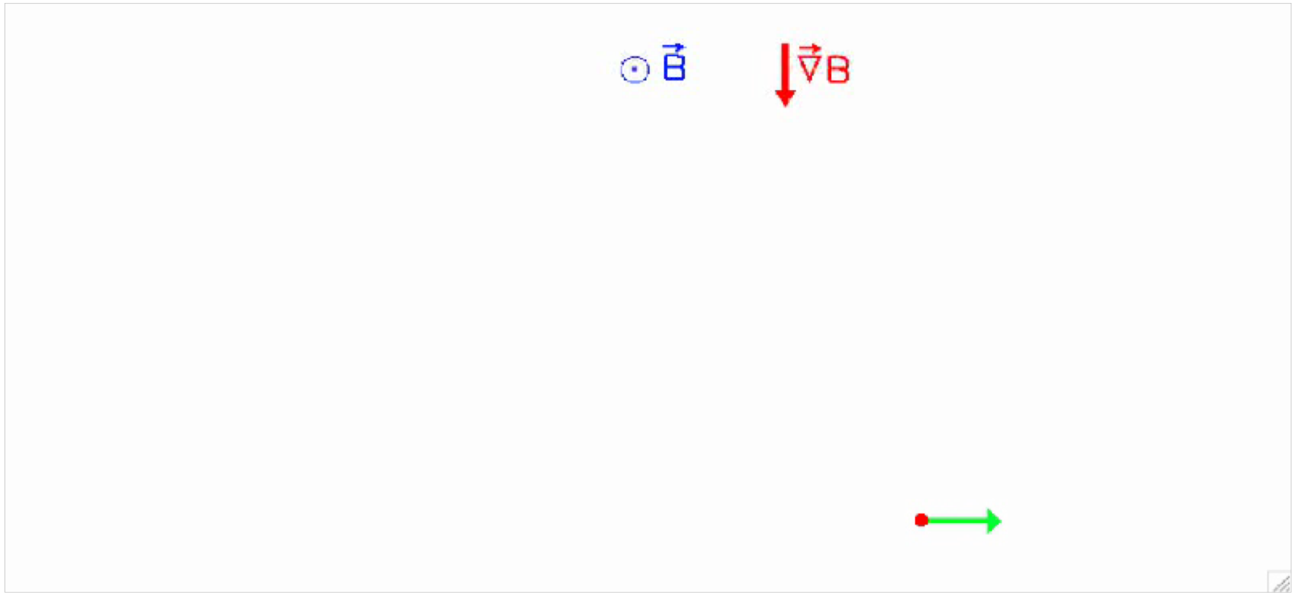


Non-uniform B-field

•One can derive this drift by Taylor expanding the B-field, taking advantage of the smallness of the Larmor radius (keeping only 1st order terms)

$$B_z(x,y,z) = B_0 + \frac{\partial B_z}{\partial y}(y - y_{gc}) + O(\varepsilon^2)$$

$$\varepsilon = \left|\frac{\nabla B}{B}\right| r_L << 1$$



Adiabatic invariants

- Instead of Taylor expanding, it is also possible to derive this drift much faster by introducing the first adiabatic invariant μ:
  - This invariant is very useful in several contexts
- Background:
  - The concept of adiabatic invariants is known from analytic mechanics
  - Assume a particle performs periodic motion in one coordinate *q*
  - Then one can define the action as: $\oint p_q dq$
    - Here p_q is the generalized momentum associated with *q*
  - If one perturbs the periodic motion by a small amount ε, the action remains conserved, to all powers in ε
- We have already one periodic motion – the gyration. The coordinate for the gyration is θ, and $p_\theta = mv_\theta r$ is the associated generalized momentum (we recognize it's just the angular momentum in the gyration)

The first adiabatic invariant

$$\oint p_q dq = \int_0^{2\pi} mv_\theta r d\theta = \int_0^{2\pi} mv_\perp r_L d\theta = 2\pi mv_\perp r_L =$$

$$2\pi mv_\perp \frac{mv_\perp}{qB} = \frac{4\pi m}{q} \frac{\frac{1}{2}mv_\perp^2}{B} = \text{constant}$$

$$\text{so } \mu = \frac{\frac{1}{2}mv_\perp^2}{B} = \text{constant}$$

- μ is conserved – it is actually the magnetic dipole moment of the charged particle, if we consider the particle as a charged current ring with radius r_L

$$IA = \frac{q\omega_c}{2\pi}\pi r_L^2 = \frac{q^2 B}{2\pi m}\pi\left(\frac{mv_\perp}{qB}\right)^2 = \frac{mv_\perp^2}{2B} = \mu$$

The first adiabatic invariant is the dipole moment

$$IA = \frac{q\omega_c}{2\pi}\pi r_L^2 = \frac{q^2 B}{2\pi m}\pi(\frac{mv_\perp}{qB})^2 = \frac{mv_\perp^2}{2B} = \mu$$

•This magnetic dipole is anti-aligned with the magnetic field (a plasma is diamagnetic)

•A magnetic dipole with strength μ embedded in a magnetic field B anti-aligned to the dipole has potential energy μB, so it feels a force

This is the so-called mirror force $\vec{F} = -\mu\nabla B$

This force also works for neutral particles as long as they have a magnetic dipole moment (Example: Antihydrogen)

If the force is along B, it can provide some confinement along the field lines for charged particles – they can be reflected by a magnetic mirror

(Mirror confinement was attempted for fusion but is not pursued much these days)

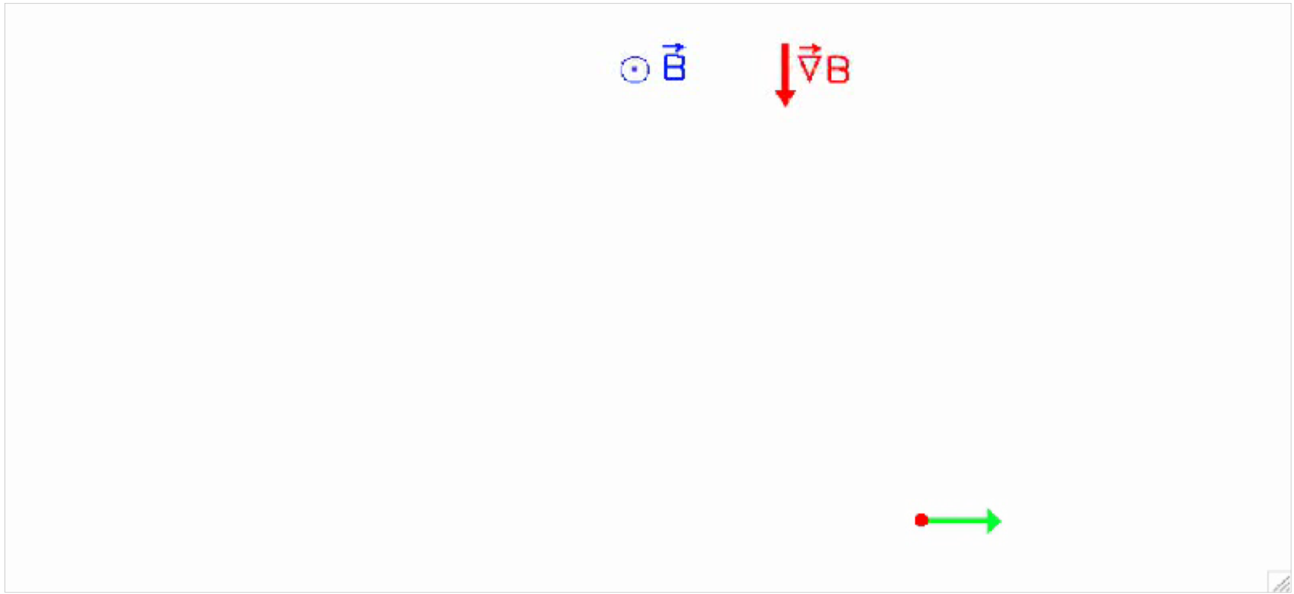
If the force is perpendicular to B, then we get a drift

Non-uniform B-field strength

- We use now the FxB formula we derived earlier:

$$\vec{F} = -\mu \nabla B = -\mu \frac{dB_z}{dy} \hat{y}$$

$$\vec{v}_{\nabla B} = \frac{\vec{F} \times \vec{B}}{qB^2} = \frac{-\mu \nabla B \times \vec{B}}{qB^2} = \frac{mv_\perp^2}{2B} \frac{\vec{B} \times \nabla B}{qB^2}$$



Non-uniform B-field direction: Curvature drift

- If the magnetic field strength is inhomogeneous, the magnetic field is usually also curved
    - It has to be curved if it's inhomogeneous, unless you have significant currents
- In the guiding center approximation, the zeroth order motion is along the magnetic field.
- So if the magnetic field is curved, the particle feels a centrifugal force:

$$\vec{F}_C = \frac{mv_\parallel^2}{R_C}\frac{\vec{R}_C}{R_C}$$

$$\vec{v}_{R_C} = \frac{\vec{F}_C \times \vec{B}}{qB^2} = \frac{mv_\parallel^2}{R_C^2}\frac{\vec{R}_C \times \vec{B}}{qB^2}$$

Combining the two drifts

- With a bit of algebra, we can combine the grad B and curvature drifts into one formula – assuming that the current density is negligible. This is not universally true but often enough that it is useful to derive this combined formula.

$$\nabla \times \vec{B} = \mu_0 \vec{j} = 0$$

$$\nabla \cdot \vec{B} = 0$$

- If the magnetic field has curvature, we can go into a local cylindrical coordinate system with the axis given by the axis for the radius of curvature (blackboard). In that coordinate system:

$$\vec{B} = B_r(r,\theta,z)\hat{r} + B_\theta(r,\theta,z)\hat{\theta} + B_z(r,\theta,z)\hat{z} = B_\theta(r,\theta,z)\hat{\theta} = B_\theta(r,\theta)\hat{\theta}$$

Combining the two drifts

$$\vec{B} = B_r(r,\theta,z)\hat{r} + B_\theta(r,\theta,z)\hat{\theta} + B_z(r,\theta,z)\hat{z} = B_\theta(r,\theta,z)\hat{\theta} = B_\theta(r,\theta)\hat{\theta}$$

- We need Goldston and Rutherford (or another formula book) to write the differential operators in cylindrical coordinates:

$$\nabla \cdot \vec{B} = 0 \Rightarrow \frac{1}{r}\frac{\partial(rB_r)}{\partial r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0 \Rightarrow \frac{\partial B_\theta}{\partial \theta} = 0 \Rightarrow \vec{B} = B_\theta(r)\hat{\theta}$$

- This helps us significantly simplify the curl equation, which is otherwise terribly complicated:

$$\nabla \times \vec{B} = \frac{1}{r}\left(\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (rB_\theta)}{\partial r} - \frac{\partial B_r}{\partial \theta}\right)\hat{z}$$

$$\nabla \times \vec{B} = \frac{1}{r}\left(\frac{\partial (rB_\theta)}{\partial r}\right)\hat{z} = 0 \Leftrightarrow rB_\theta = \text{constant (c)}$$

$$\Leftrightarrow B_\theta = \frac{c}{r}$$

Combining the two drifts

$$B = B_\theta = \frac{c}{r} \Rightarrow \nabla B = -\frac{c}{r^2} r$$

$$\frac{\nabla B}{B} = \frac{-\frac{c}{r^2}\hat{r}}{\frac{c}{r}} = \frac{-\hat{r}}{r} = \frac{-\vec{R}_c}{R_c^2}$$

Combining the two drifts

- Thus, we have shown in the last few slides that:

$$\left.\begin{array}{r}\nabla\times\vec{B}=\mu_0\vec{j}=0\\ \nabla\cdot\vec{B}=0\end{array}\right\}\Rightarrow\frac{\nabla B}{B}=-\frac{\vec{R}_C}{R_C^2}$$

- Therefore we can combine the two magnetic non-uniformity drifts into one formula:

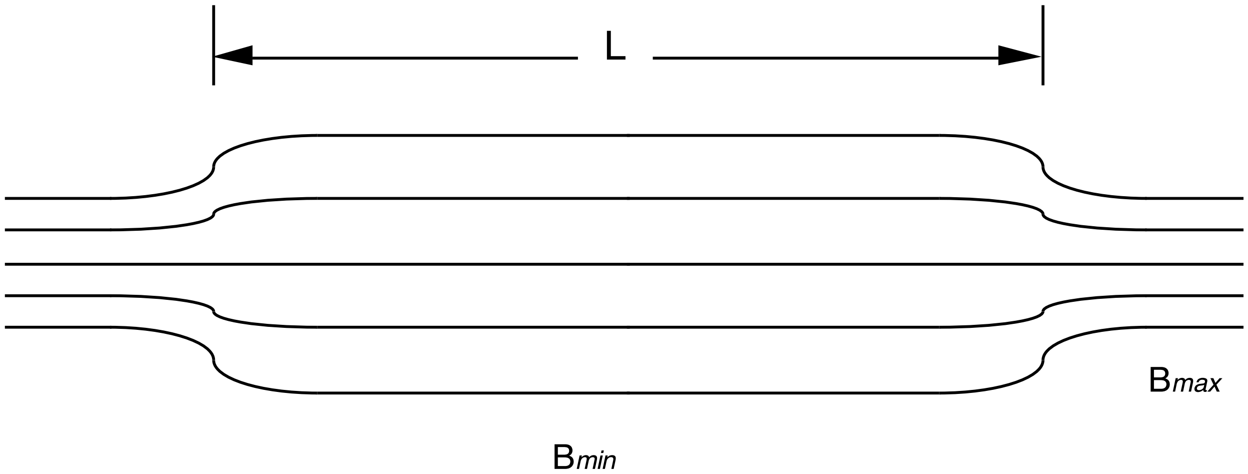
$$\vec{v}_{\nabla B+R_C}=\frac{2mv_{\parallel}^2+mv_{\perp}^2}{2B}\frac{\vec{B}\times\nabla B}{qB^2}$$

- These two drifts add up – no magic cancellation
- The drifts are “slow” in general, verifying our perturbative approach:
  - E=100 eV electron in B=1 T and R_C= 1 m:
    - v_{Rc}~100 m/s vs ~5*10^5 m/s free streaming velocity and r_L ~3 µm

Mirror force – mirror confinement

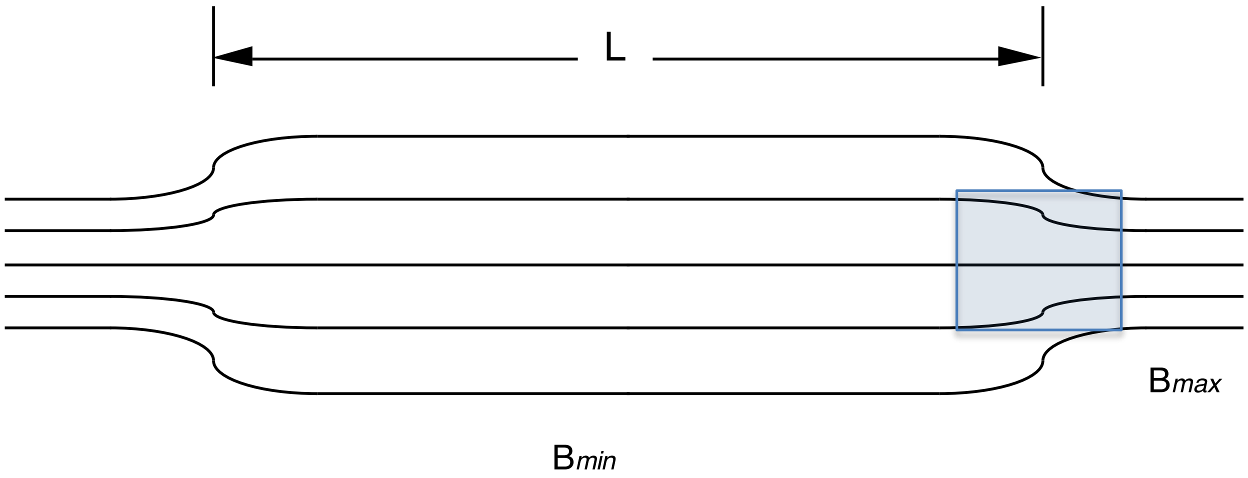
•The mirror force $\mu \nabla B$ for a guiding center particle can also be parallel to B, and thereby provide confinement of guiding center particles in the third direction

•This immediately brings up the question whether this is physical – how can $\vec{v} \times \vec{B}$ have a component parallel to $\vec{B}$?



- Look at a particle whose guiding center is on the axis – the straight field line - of the magnetic configuration above

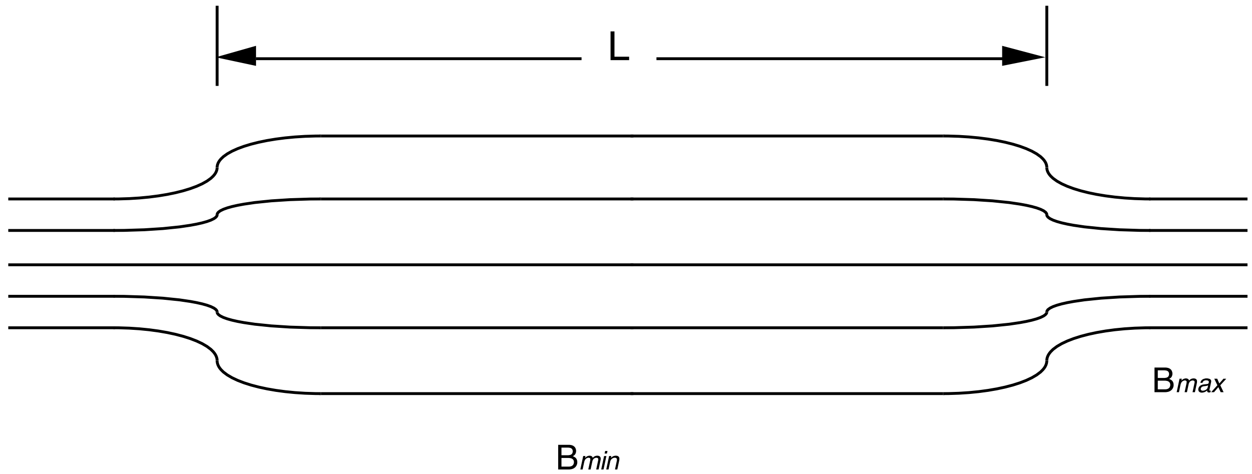
Mirror force for a generic situation



- The particle is in in a region of converging field lines:

$$B_z = B_z(z)$$
$$\nabla \cdot \vec{B} = 0$$
$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} = 0 \Leftrightarrow \frac{\partial}{\partial r}(rB_r) = -r\frac{\partial B_z}{\partial z}$$
$$rB_r = -\frac{1}{2}r^2\frac{\partial B_z}{\partial z} \Leftrightarrow B_r = -\frac{1}{2}r\frac{\partial B_z}{\partial z}$$
$$\vec{F} = q\vec{v}\times\vec{B} \Rightarrow F_z = qv_\theta B_r \Leftrightarrow$$
$$F_z = -\frac{1}{2}qv_\theta r_L\frac{\partial B_z}{\partial z} = -\frac{1}{2}qv_\perp\frac{mv_\perp}{|q|B}\frac{\partial B_z}{\partial z} = -\mu\frac{\partial B_z}{\partial z}$$

Mirror confinement (mirror-reflected particles)



$$\mu B = \frac{1}{2} m v_\perp^2, \mu = constant$$

- Now that we believe in the mirror force, we see that it can be used for confinement.
- We also recognize that μB is the potential energy of the guiding center particle due to its magnetic moment.
- A guiding center particle is in a potential well in a mirror device (above)
- For which particles is the potential well sufficiently deep for trapping?

Mirror confinement (mirror-reflected particles)

$$\frac{\partial \vec{B}}{\partial t} = 0$$

$$\mu B = \frac{1}{2} m v_\perp^2, \mu = constant$$

$$\frac{1}{2} m v^2 = \frac{1}{2} m v_\perp^2 + \frac{1}{2} m v_{||}^2 = \mu B + \frac{1}{2} m v_{||}^2 = \text{constant} = E_k$$

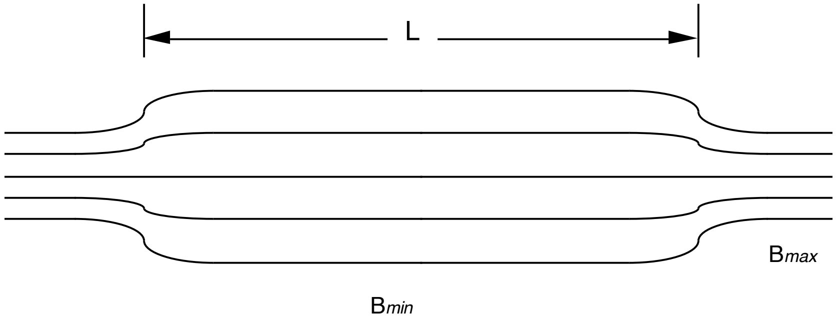
$$\mu B \leq E_k$$

If $\mu B_{max} > E_k$ then the particle is trapped, otherwise it passes through

$$\mu = \frac{1}{2} m v_{\perp 0}^2 / \mathrm{B_{min}}$$

$$\frac{1}{2} \frac{m v_{\perp 0}^2}{B_{min}} B_{max} > \frac{1}{2} m v_0^2 \Leftrightarrow \frac{B_{max}}{B_{min}} > \frac{v_0^2}{v_{\perp 0}^2}$$

Mirror confinement (mirror-reflected particles)



$$\frac{\partial \vec{B}}{\partial t} = 0$$

$$\frac{B_{max}}{B_{min}} > \frac{v_0^2}{v_{\perp 0}^2} = \frac{1}{\sin^2\theta} \Leftrightarrow \sin^2\theta > \frac{B_{min}}{B_{max}}$$

- θ is the angle between the velocity vector and the B-field vector
- Only part of the Maxwellian is confined – there are always particles that have a small pitch angle
- The mirror ratio B_{max}/B_{min} is limited by available magnet technology and by needing a minimum (!) B_{min} so the particles are confined in the long region
- The trapped particles perform a periodic motion in the parallel direction giving rise to a second adiabatic invariant J=$\oint_0^L v_{||}dz$
- An electrostatic potential can add (or subtract) to the potential well: $q\phi + \mu B$
- For uniform B (mirror ratio 1) we must use an electrostatic potential: Penning trap

The parallel Larmor radius

- For the magnetic moment to be conserved, the gyration needs to be a nearly periodic motion. We earlier required that the Larmor radius be small compared to the distance over which the B-field changes:

$$\frac{B}{|\nabla B|} >> r_L = \frac{mv_\perp}{qB}$$

- We clearly must require also that the particle does not move into a region with a substantially different magnetic field (direction or strength) in a single gyration.

$$\frac{B}{|\nabla B|} >> \frac{2\pi v_\parallel}{\omega_c} = 2\pi \frac{mv_\parallel}{qB} = 2\pi r_{L\parallel}$$

- Thus, the "parallel Larmor radius" must also be small compared to the characteristic scale length over which B changes

Time varying fields

- One can show that μ is conserved also when dB/dt is nonzero, as long as the B field variation is small in one gyration and has no frequency component near ω_c
  - Implies that perpendicular kinetic energy changes, which is due to E:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- A time varying E-field gives rise to a so-called polarization drift (in addition to an altered ExB drift):

$$\vec{v}_D = \frac{m \, \partial \vec{E} / \partial t}{qB^2} \text{ if } \omega << \omega_c$$

- For $\omega=\omega_c$ there is no μ conservation, and the polarization drift formula is no longer valid:
  - Cyclotron resonance (can be used for plasma heating for example)

Summary

- In order to confine charged particles in a magnetic field for many single particle collisions:
    - Must have a small Larmor radius
- In this limit, full orbit calculations are very expensive, and generally not necessary
- The guiding center approximation is very useful here:
    - Charged gyrating particle is approximated as a charged current ring, with constant anti-aligned magnetic dipole moment, sliding along the magnetic field line in the ring center
    - Such particles have slow drifts away from their “birth” magnetic field line, which can be calculated analytically
    - Can calculate complicated trajectories with relative ease
    - Can identify good magnetic traps (particles confined for many collisions) and ones that are less successful…

- Can identify good magnetic traps (particles confined for many collisions) and ones that are less successful…

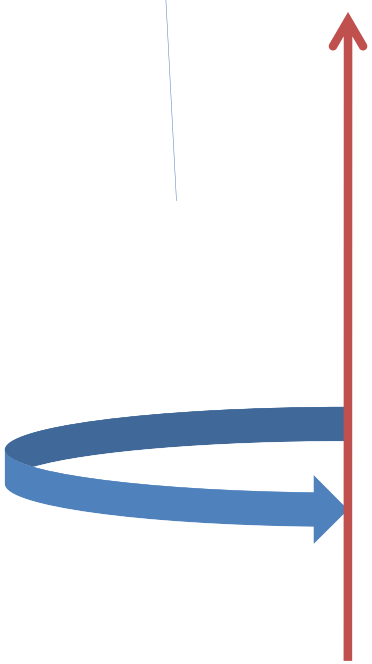
Examples in the following slides:

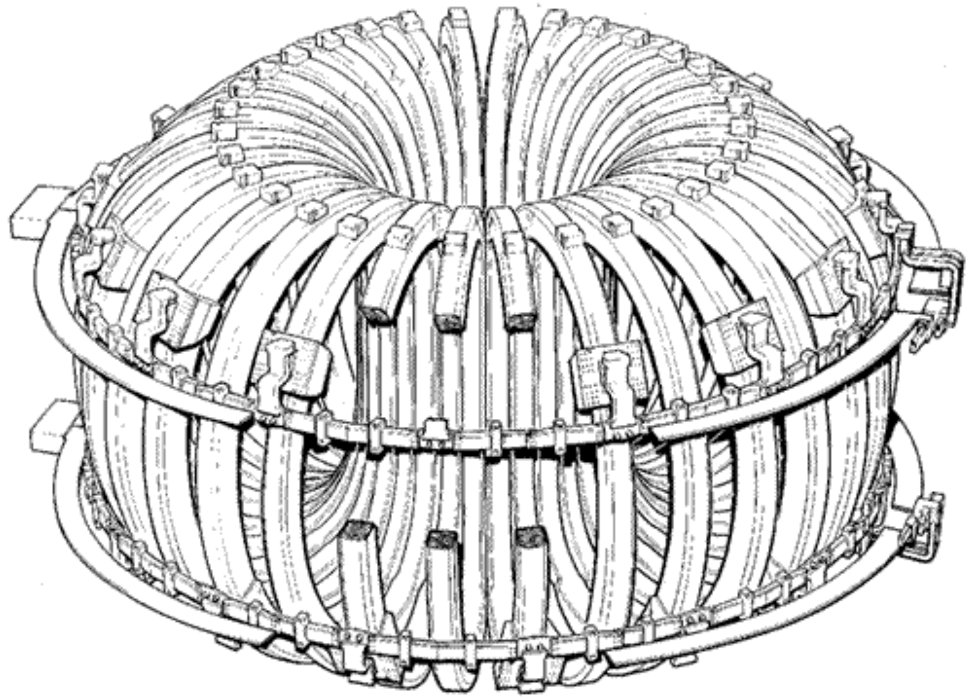
Pure toroidal field trap (Example:neutral plasma)

- The magnetic field from a dense toroidal set of coils is equivalent to that from an infinite, straight current carrying wire
- Particles experience magnetic drifts in the vertical direction:

$$\vec{v}_{\nabla B + R_C} = \frac{2mv_{\parallel}^2 + mv_{\perp}^2}{2B} \frac{\vec{B} \times \nabla B}{qB^2}$$

Opposite for ions and electrons – they drift apart





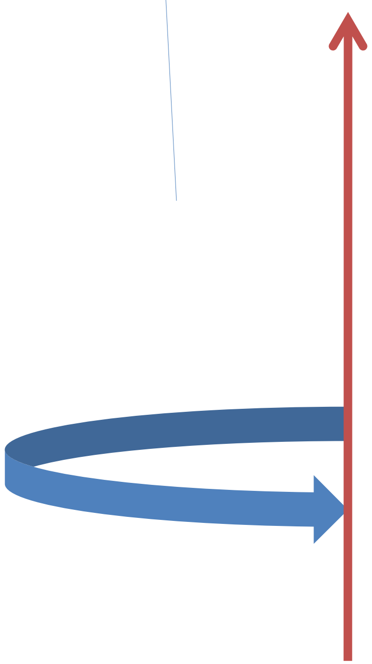
Pure toroidal field trap (Example:neutral plasma)

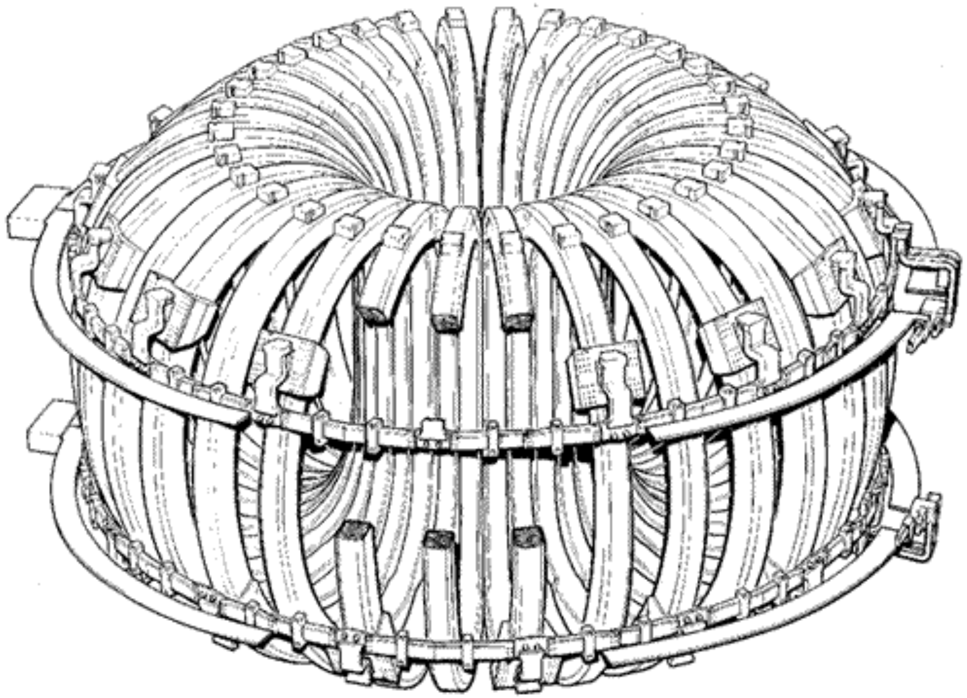
- Opposite for ions and electrons – they drift apart

$$\vec{v}_{\nabla B + R_C} = \frac{2mv_{\parallel}^2 + mv_{\perp}^2}{2B}\frac{\vec{B}\times\nabla B}{qB^2}$$

- Vertical electric field sets up: $\vec{v}_E = \frac{E\hat{z}\times B\hat{\varphi}}{B^2} = \frac{E}{B}\hat{R}$

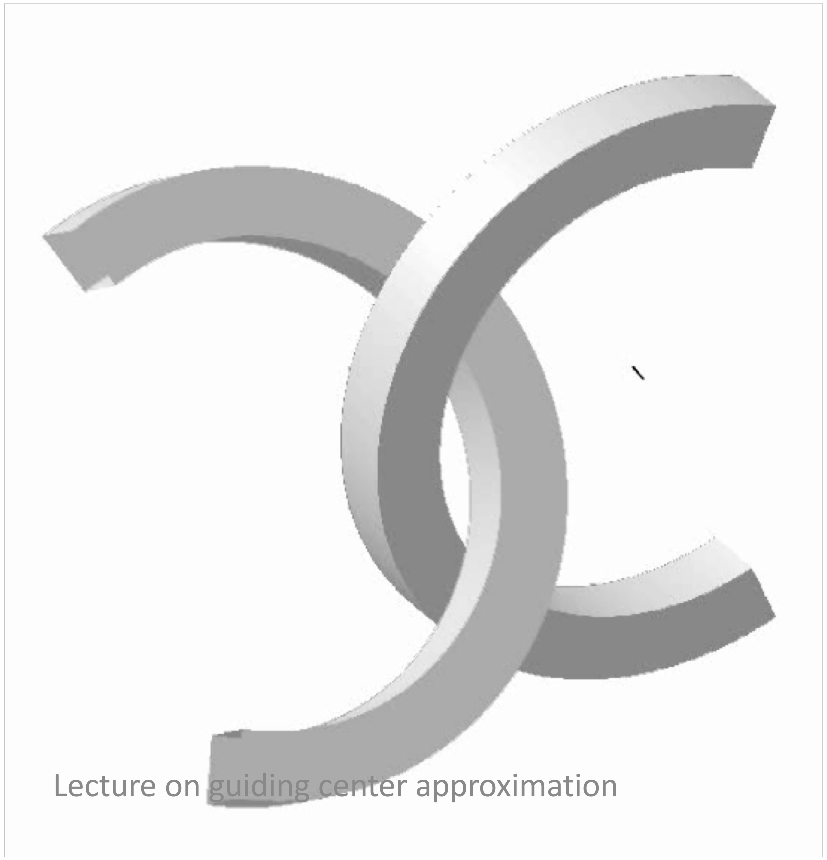
- Outward radial expansion of plasma – no confinement





Non-neutral plasmas in a stellarator

- A stellarator is a magnetic surface configuration: Each magnetic field line wraps around a toroidal surface, never leaving the surface.
- Also mainly toroidal field – also vertical drift of particles?
- No – vertical drifts cancel because of the poloidal motion that the particle has, as a result of parallel motion along the magnetic field

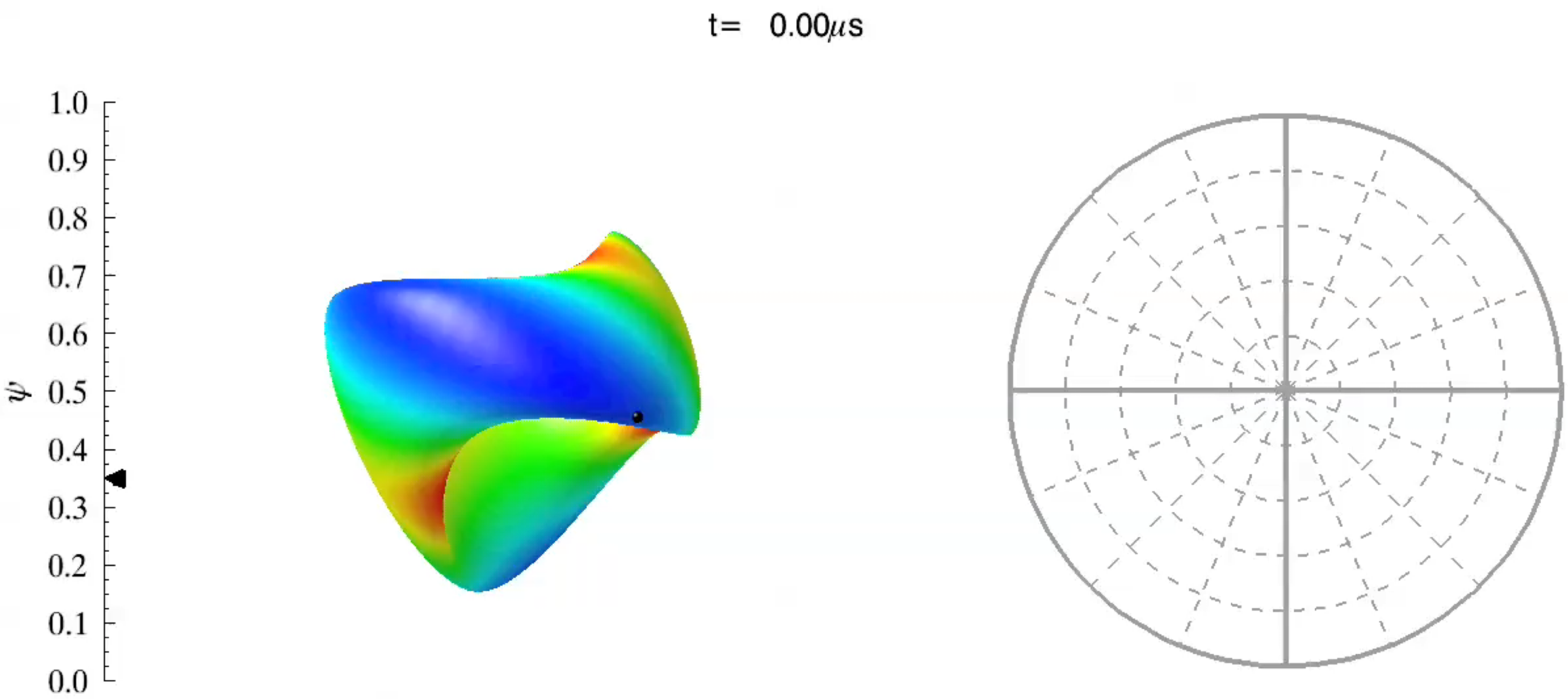


Without E-field, CNT has “bad” orbits!

CNT is a “classical stellarator” – will not work well for fusion:

About 50% of particles are magnetically trapped (due to mirror force/first adiabatic invariant).

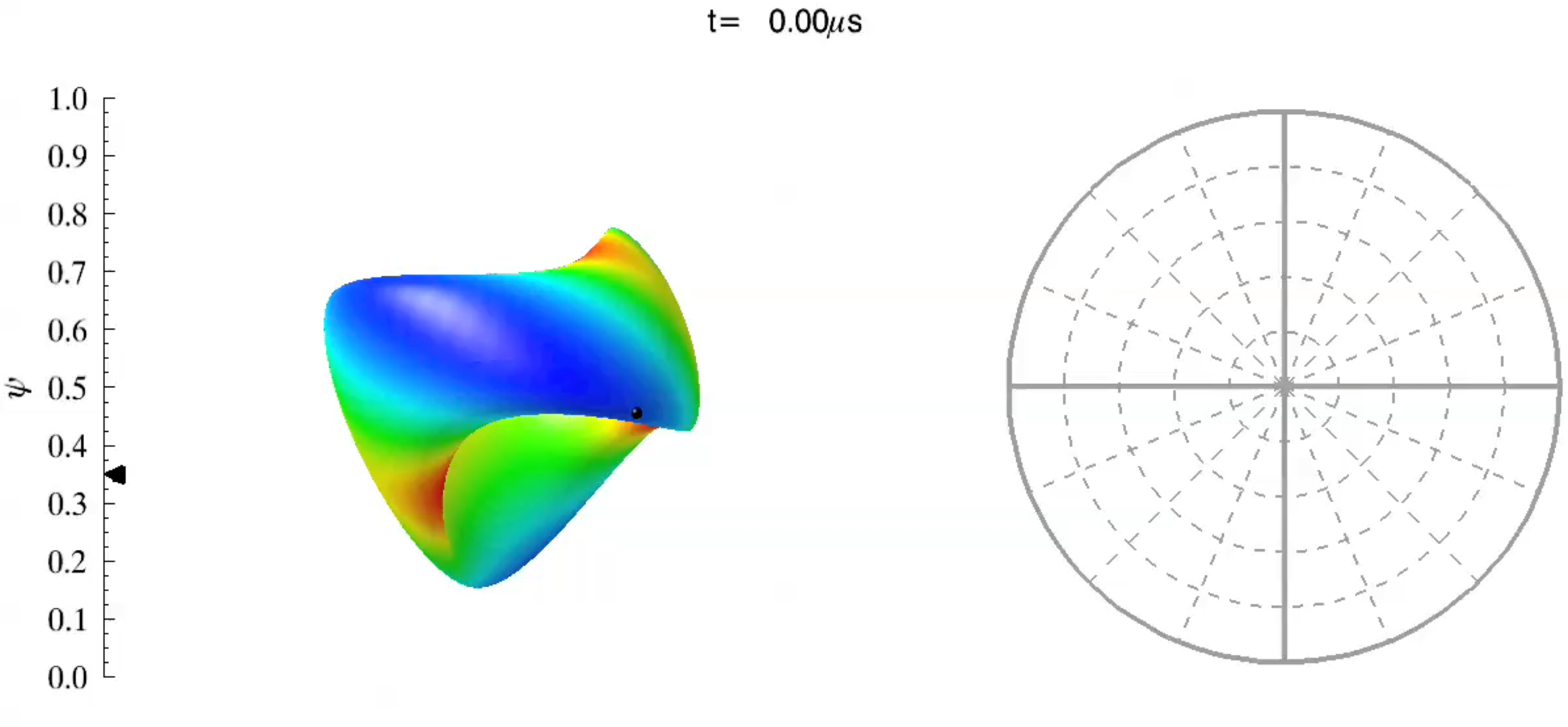
They don’t circulate toroidally, therefore don’t circulate poloidally, and drift out of CNT due to the magnetic drifts. Example:



ExB could come to the rescue

A strong space charge electric field – constant on a magnetic surface – is added to the simulation of the trapped particle

Now it is confined! For much the same reasons as in the pure toroidal field trap



Electric fields in stellarators

How large of a role does the bulk ExB drift play relative to the magnetic drifts?

$$\left|\frac{v_{ExB}}{v_{\nabla B}}\right| \approx \left|\frac{\nabla\phi / B}{(W_k \nabla B / eB^2)}\right| \approx \left|\frac{e\phi}{W_k}\right|$$

Pure-electron plasma: Dominant (factor of 10-1000)

Thermal particles in a quasineutral plasma: Depends.. (0.2-5)

Set by ambipolarity: In steady state, the positive and negative charges of the plasma must leave at the same rate (they “arrive” by neutralization of atoms – ie. at the same rate).

If one species has a tendency to be less well confined (higher mass, higher temperature etc) it will initially leave faster, leaving space charge electric fields that begin to hold it back.

For some plasmas, T_e>>T_i which drives a relatively strong positive radial electric field to “hold electrons in and push ions out” – but actually typically improves confinement of both species

For Te~Ti, usually a negative radial electric field develops

The orbit healing magic of a radial electric field cannot “fix” α-confinement in a future reactor: Ratio is negligibly small: ~35 keV/3.5 MeV~0.01)